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Evaluation of Selected Finite Difference Solutions to the Shallow Water Equations

Algorithms, Fortran code and examples

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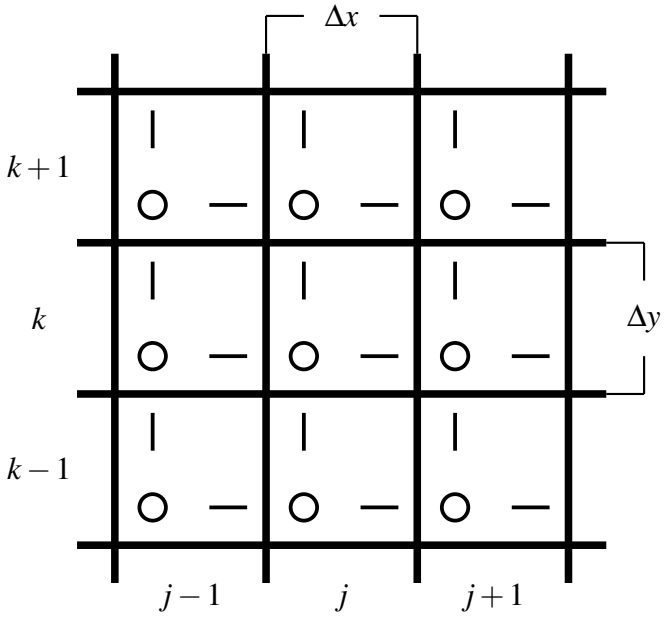

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Abstract

Considered are three different finite difference schemes to solve the rotating, two-dimensional shallow water equations. These are the linear forward-backward scheme (FB-L), the linear centered in time, centered in space scheme (CTCS-L) and a nonlinear version of the latter (CTCS-N). The resulting algorithms are programmed using the FORTRAN 95 programming language, and compiled and run on a PC using the PC's central processing unit (CPU). The schemes are tested for four cases, the breaking of a dam on a wet domain, geostrophic adjustment, small Kelvin waves and large Kelvin waves. The solutions are compared with results from identical cases derived by *Holm et al.* (2020), who in addition to the above schemes also included two modern finite volume schemes. It is undescored that while this work make use of the computer's CPU, *Holm et al.* (2020) utilized the computer's graphical processor unit (GPU). The latter necessitates the use of a GPU programming language. It is therefore of interest to investigate whether the results derived here are equal to those of *Holm et al.* (2020). For all cases the two *linear* schemes provide results nearly identical to the results derived by *Holm et al.* (2020) using their linear scheme. In contrast the solutions provided by the *nonlinear* CTCS-N scheme differ radically from those derived by *Holm et al.* (2020) for two of the cases, namely the breaking on a wet domain and the large Kelvin wave cases. Interestingly the CTCS-N solution derived here is more in line with what one would expect in terms of the physics involved, and also more in line with the solutions provided by the two finite volume schemes of *Holm et al.* (2020).

Keywords

Oceanography, Numerical Modeling, Shallow Water Equations

Disciplinary signature	Responsible signature
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Contents

1 Introduction

Considered are solutions to the rotating shallow water equations utilizing three different numerical finite difference schemes. To this end four of the cases presented by *Holm et al.* (2020) are revisited, namely

- Case I: Dam break on a wet domain,
- Case II: Rossby adjustment,
- Case III: Small Kelvin waves
- Case IV: Large Kelvin waves.

Case I corresponds to Case A of *Holm et al.* (2020), Case II to Case B and Cases III and IV to the small and large Kelvin waves cases included in their Case D.

All solutions are derived on an Arakawa C-grid (Lattice C of *Mesinger and Arakawa*, 1976; *Røed*, 2019) using three somewhat different schemes. These are

- FB-L: the Forward-Backward Linear scheme,
- CTCS-L: the linear version of the traditional Centered in Time, Centered in Space scheme (CTCS), also widely known as the leapfrog scheme,
- CTCS-N: the nonlinear version of the CTCS scheme

The FB-L scheme, which is identical to the linear scheme studied by *Holm et al.* (2020), is credited to *Sielecki* (1968). It was for instance used by *Martinsen et al.* (1979) in an early study of storm surges along the western coast of Norway. The nonlinear CTCS-N scheme is the traditional choice for solving the shallow water equations on a rotating frame on an Arakawa C-grid within the atmospheric and oceanographic communities. Furthermore, it is commonly one of the options offered in Numerical Weather Prediction (NWP) and Numerical Ocean Weather Predicition (NOWP) models. It is also identical to the nonlinear scheme studied by *Holm et al.* (2020).

Since the cases and the two of the schemes (FB-L and CTCS-N) studied were also investigated by *Holm et al.* (2020), the solutions derived using these two schemes are compared to the solutions obtained by them. The CTCS-L scheme, which was not studied by *Holm et al.* (2020), is added merely to check the linear solution derived using the FB-L

scheme. Interestingly *Holm et al.* (2020) also included two additional nonlinear schemes, namely the KP scheme (credited to *Kurganov and Petrova*, 2007) and CDKLM scheme (credited to *Chertok et al.*, 2018). These are two more modern schemes developed, among other things, to handle shocks. Hence they are expected to produce decent results even for Cases I and IV which include shocks. It is underscored that while this study uses the computer's Central Processing Unit (CPU), *Holm et al.* (2020) uses the computer's Graphical Processing Unit (GPU). It is therefore of interest to compare the results derived here with those of *Holm et al.* (2020).

Below Section 2 gives a brief introduction to the nonlinear and linear shallow water equations followed by a brief outline of the continuum boundary conditions (Section 3). Details about the model ocean design and grid configuration are revealed by Section 4. Sections 5 provide details on the finite difference analogue of the governing equations for each of the three numerical schemes, and the finite difference form of the boundary conditions is treated in Section 6. Section 7 provides details about the four test cases, while Section 8 presents and discusses the results. Finally Section 9 offers a summary and some final remarks. An Appendix A is added to present the somewhat lenghty Fortran program used to solve the governing equations.

2 Governing Equations

The non-linear, rotating shallow water equations written in flux form, which may be found in almost any textbook on oceanography (e.g., *Røed*, 2019, eqs. 1.26 and 1.27 on page 7), are

$$\partial_t \mathbf{U} = -\underbrace{f\mathbf{k}\times\mathbf{U}}_{\text{Coriolis}} - \underbrace{gh\nabla_H\eta}_{\text{Pressure}} - \underbrace{\nabla_H\cdot\left(\frac{\mathbf{UU}}{h}\right)}_{\text{Advective flux}} + \underbrace{\mathbf{X}}_{\text{Forcing}}, \tag{1}$$

$$\partial_t h = -\underbrace{\nabla_H\cdot\mathbf{U}}_{\text{Divergence}}, \tag{2}$$

where

$$h(x,y,t) = \eta(x,y,t) + H(x,y), \tag{3}$$

is the thickness of a water column, η being the deviation of the surface away from its equilibrium depth H as illustrated in Figure 1, f is the Coriolis parameter, g denotes the the gravitational acceleration, and $\mathbf{X}$ includes all external and internal forcing. Furthermore,

$$\mathbf{U} = \int_{-H}^{\eta} \mathbf{u}\,dz \tag{4}$$

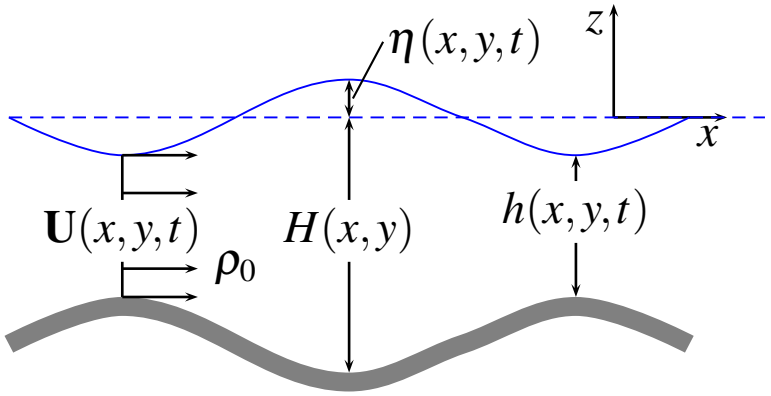

Figure 1: Sketch of a one layer, barotropic model conveniently showing some of the notation. Note that the total column height is $h = \eta + H$, so that $\eta = h - H$.

is the volume transport (hereafter transport) where $\mathbf{u}(x,y,z,t)$ is the horizontal velocity with components u, v in the x, y directions, respectively.

The various terms on the right-hand side of (1) and (2) all give rise to a time rate of change in the transport or the thickness. They are respectively

- Coriolis: containing the effect of the Earth's rotation,
- Pressure: a force exerted by the pressure gradients,
- Advective flux: responsible for advecting the transport from one location to another, and is a nonlinear term,
- Forcing: containing forces such as wind forcing, atmospheric pressure, friction along the bottom and internal mixing processes,
- Divergence: containing transport gradients that may cause the sea level to rise (convergence) or fall (divergence) at a particular location.

As is evident the pressure forcing and the advective flux terms contain nonlinear terms. Furthermore, when the Coriolis force balances the pressure gradient a state of geostrophic balance is obtained. This is a second possible steady state (cf. *Rossby*, 1937, 1938) and differs from the state at rest in that it contains both kinetic and potential energy. Last but not least many proceses in the ocean may be studied using these barotropic shallow water

equations. Prime examples are tidal motion and storm surges[1].

As mentioned the forcing term includes wind traction and atmospheric pressure forcing in addition to mixing. It therefore takes the form

$$\mathbf{X} = \mathbf{E}_v + \frac{\tau^s - \tau^b}{\rho_0} + \nabla_H p_a \tag{5}$$

where $\mathbf{E}_v$ is the eddy viscosity (internal mixing term), τ^s is the winds stress due to the wind's traction on the surface, τ^b is the bottom stress due to friction at the bottom, and $\nabla_H p_a$ is the atmospheric pressure forcing. In contrast to the eddy viscosity the latter three are all external forcing terms. They may therefore easily be added later. Henceforth only mixing is included as a forcing term in what follows. The forcing term $\mathbf{X}$ therefore reduces to

$$\mathbf{X} = \mathbf{E}_v. \tag{6}$$

Since the governing equations are non-linear, the scheme may become numerically unstable due to nonlinear instability (*Phillips*, 1959; *Røed*, 2019, chap. 10.3, page 217). Here the mixing term comes in handy and may be used to squelch the nonlinear instability. It then becomes an artificial or *numerical* mixing term. Commonly it is then parameterized as a diffusive flux, that is,

$$\mathbf{E}_v = \nabla_H \cdot \mathscr{F}; \quad \text{where} \quad \mathscr{F} = \mathscr{A} \cdot \nabla_H \mathbf{U}, \tag{7}$$

where $\mathscr{F}$ is the eddy viscosity flux and $\mathscr{A}$ is the eddy viscosity (a tensor). Since the mixing term is artificial, it is usually parameterized so that

$$\mathscr{A} = \begin{bmatrix} A & 0 & 0 \\ 0 & A & 0 \\ 0 & 0 & A \end{bmatrix} \tag{8}$$

where A is a constant. Under these circumstances the eddy viscosity takes the final form

$$\mathbf{E}_v = A\nabla_H^2 \mathbf{U}. \tag{9}$$

3 Boundary and initial conditions

To solve (1) and (2) boundary and initial conditions are needed to determine the integration constants. Examination of (1) and (2) reveals that they are of second order in space.

[1] In many cases even the linear solution to the shallow water equations is a good approximation to simulate tides and storm surges.

Thus eight boundary conditions are allowed, four in x and four in y. In time only three initial conditions are allowed. If more are specified the system is overdetermined in a mathematical sense, which may or may not lead to false solutions. Regarding the CTCS schemes this leads to the well known initial boundary value problem as for instance explained by *Røed* (2019) (Chapter 5.6, page 84).

The conditions to be applied at the boundaries are either demanded by the physics of the problem or, in the case of open boundaries, must be constructed (e.g., *Chapman*, 1985; *Røed and Cooper*, 1987; *Palma and Matano*, 2000; *Røed*, 2019). In the cases detailed in Section 7 the model ocean is contained within a rectangular basin oriented so that the boundaries are aligned with the north-south and east-west directions. Furthermore, the boundaries to the north and south are closed boundaries while the boundaries to the west and east are open. Thus, both open and closed boundary conditions must be imposed.

3.1 Closed boundary conditions

At a closed boundary the transport components normal to and along the boundary is required to satisfy the so called no-slip condition. Hence

$$\mathbf{U} = 0, \quad \text{at} \quad S, \tag{10}$$

where S is the boundary. At the southern boundary of the domain, which is a straight, coastal wall along $y = 0$, the no-slip condition takes the form

$$U = V = 0, \quad \text{at} \quad y = 0, \quad \forall x. \tag{11}$$

Likewise along the northern boundary $y = L_y$ it takes the form

$$U = V = 0, \quad \text{at} \quad y = L_y, \quad \forall x. \tag{12}$$

3.2 Open boundary conditions

In Cases I and II the Flow Relaxation Scheme (FRS *Røed*, 2019, Chapter 7.5) is utilized as an open boundary condition at the eastern and western boundaries. In contrast a cyclic or periodic condition (*Røed*, 2019, Chapter 2.4, page 18) is applied in Cases III and IV.

A cyclic boundary condition is imposed by letting

$$U, V, h(x + L_x, y, t) = U, V, h(x, y, t). \tag{13}$$

where L_x is the length of the basin in the eastern direction.

The FRS, which was originally developed by *Davies* (1976) for the atmosphere and later adapted for ocean modeling by *Engedahl* (1995a), is a predictor-corrector scheme so that

$$\psi_{corr} = (1-\alpha)\,\psi_{pred} + \alpha\,\psi_{ext}\,. \tag{14}$$

Here ψ_{corr} is the corrector and ψ_{pred} is the predictor, while ψ_{ext} is a specified external solution. The coefficient α is such that $0 \leq \alpha \leq 1$ within the FRS zones at the eastern and western end of the computational domain and zero elswhere. Thus, $\psi_{corr} = \psi_{ext}$ at the end of the computational domain and $\psi_{corr} = \psi_{pred}$ in the interior domain. In this respect it is important that the gradient of α is as close to zero as possible at the boundary between the FRS zone and the interior domain where $\alpha = 0$. To achieve this the FRS zone has to be sufficiently wide, say $\sim$10 grid points.

4 Model ocean and grid configuration

The model ocean is the same as used by *Holm et al.* (2020). It is a rectangular basin oriented so that the x-axis of a Cartesian coordinate system points eastward and the y-axis northwards (Figure 2). Consequently the four boundaries are referred to as the northern (top), southern (bottom), western (left-hand) and eastern (right-hand) boundary, respectively. At these boundaries the solution is determined by the boundary conditions. The size of the domain may be different from case to case (cf. Table 1 on page 23).

In all the four cases considered the northern and southern boundaries are closed while the western and eastern boundaries are open. At the former boundaries the no-slip condi-

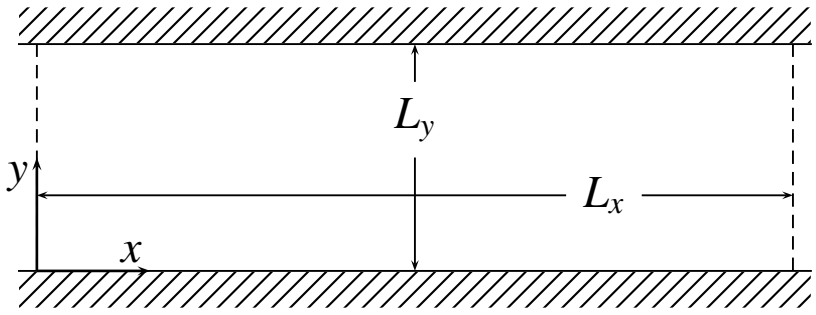

Figure 2: Sketch of the model ocean domain used when performing the cases. The basin is oriented so that the x-axis points towards the east in a Cartesian coordinate system. The dashed lines indicate open boundaries, while the solid hatched lines indicate closed, reflective walls.

tion is imposed (Section 3.1). At the open boundaries either a cyclic boundary condition (Cases III and IV) or the Flow Relaxation Scheme (Cases I and II) (Section 3.2) is imposed.

The grid configuration used to construct the finite difference approximation (FDA) to (1) and (2) is as outlined by Figure 3. It corresponds to Lattice C of *Mesinger and Arakawa* (1976). The solid blue lines conform with the boundaries of the model ocean shown by Figure 2. As is evident the model ocean boundaries are aligned along the auxiliary points so that it goes through U-points along the western and eastern boundaries and through V-points along the southern and northern boundaries. Thus there are no V-points along the western and eastern boundaries, and no U-points along the northern and southern boundaries. An extra line of cells (referred to as ghost cells) outside of the model ocean domain is therefore added along the northern boundary to satisfy the no-slip boundary condition (11) at the northern wall.

The choice of a staggered C-grid is done to properly handle the boundary conditions. The Cartesian coordinate system has its origo at the auxiliary point in grid cell (1,1) as displayed by Figure 3. The staggering of the grid is such that a U-point is staggered one half grid length along the x direction with respect to an h-point, while a V-point is staggered one half grid length along the y direction with respect to an h-point. Relative to this Cartesian coordinate system the notation x_j, y_k, $U^n_{jk}, V^n_{jk}, h^n_{jk}$ entails

$$
\begin{aligned}
x_j &= (j-1)\Delta x, \quad y_k = (k-1)\Delta y, && (15)\\
h^n_{jk} &= h\left[(j-\frac{3}{2})\Delta x, (k-\frac{3}{2})\Delta y, t^n\right], && (16)\\
U^n_{jk} &= U\left[(j-1)\Delta x, (k-\frac{3}{2})\Delta y, t^n\right], && (17)\\
V^n_{jk} &= V\left[(j-\frac{3}{2})\Delta x, (k-1)\Delta y, t^n\right]. && (18)
\end{aligned}
$$

where $\Delta x, \Delta y$ are the space increments along x, y, respectively, the subscripts j,k are counters and refer to the respective cells of the C-grid holding the individual U,V,h-points in space (Figure 3). The superscript n refers to time level such that the time $t^n = n\Delta t$ where Δt is the time step.

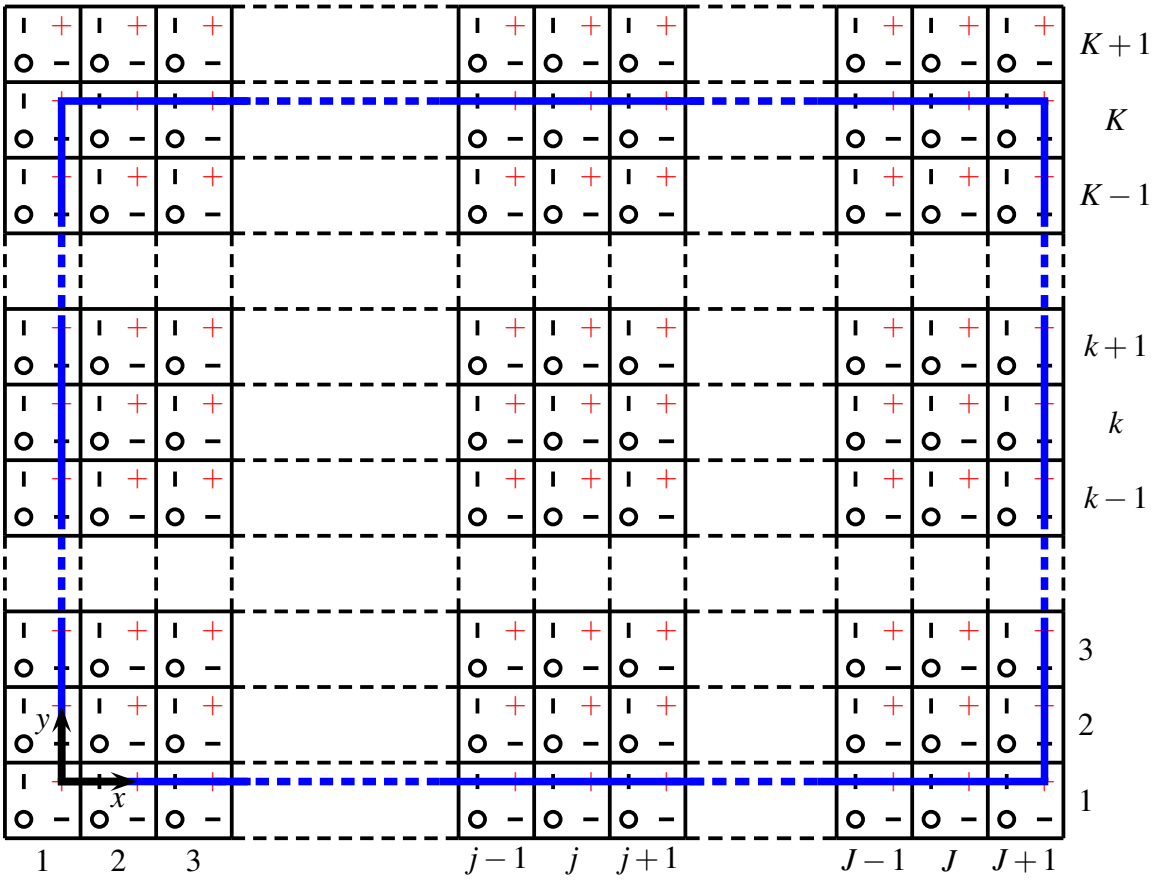

Figure 3: The staggered grid configuration used by the FB-L, CTCS-L and CTCS-N schemes corresponding to the rectangular basin shown by Figure 2. The space increments of the grid are $\Delta x, \Delta y$, while the dummy indices j,k counts the grid cells in the x and y directions, respectively. *Circles* correspond to h, η and H-points, *horizontal dashes* to U-points and *vertical dashes* to V-points. The points marked with *red plus signs* are auxiliary points. Origo in the Cartesian coordinate system corresponds to the auxiliary point located at the lower left-hand corner in cell number (1,1) where the southern and western boundaries intersects.

5 Finite difference equations

The finite difference approxomations of the three schemes listed in Section 1 may for instance be found in *Røed* (2019) (Chapters 6 and 9). For reference purposes they are nevertheless repeated here. The starting point is the governing equations (1) and (2) in their scalar form, viz.,

$$\partial_t U = fV - gh\partial_x \eta - \partial_x \left(\frac{U^2}{h}\right) - \partial_y \left(\frac{UV}{h}\right) + X_u, \tag{19}$$

$$\partial_t V = -fU - gh\partial_y \eta - \partial_x \left(\frac{UV}{h}\right) - \partial_y \left(\frac{V^2}{h}\right) + X_v, \tag{20}$$

$$\partial_t h = -\partial_x U - \partial_y V, \tag{21}$$

$$\eta = h - H, \tag{22}$$

where X_u, X_v are the forcing terms along the x-axis and y-axis, respectively. In accord with (9) they take the forms

$$X_u = A\left(\partial_x^2 U + \partial_y^2 U\right), \quad \text{and} \quad X_v = A\left(\partial_x^2 V + \partial_y^2 V\right), \tag{23}$$

respectively.

To arrive at the linear version of (19) through (22) the nonlinear, advective flux terms are discarded. Furthermore, it is assumed that the sea level deviation is small compared to the depth of a water column, so that $\eta \ll H$ everywhere. Since the mixing terms are added mainly to squelch the nonlinear instability they are superfluous as well. Furthermore, the bottom is assumed to be flat so that $H = H_0$ where H_0 is a uniform equilibrium depth. Thus, (19) through (22) take the forms

$$\partial_t U = fV - c_0^2 \partial_x \eta, \tag{24}$$

$$\partial_t V = -fU - c_0^2 \partial_y \eta, \tag{25}$$

$$\partial_t h = -\partial_x U - \partial_y V, \tag{26}$$

$$\eta = h - H_0, \tag{27}$$

where $c_0 = \sqrt{gH_0}$ is the phase speed of linear, coastally trapped Kelvin waves (cf. *Røed*, 2019, Chapter 6.3). It should be emphasized that (24) through (27) are stricly speaking only valid for linear problems and for problems where $\eta \ll H_0$ everywhere. Nevertheless, solved numerically in cases when these approximations are violated, they may still provide numerically stable solutions that may even look reasonable albeit being wrong.

5.1 The FB-L scheme

By replacing all the derivatives appearing in (24) through (27) by FDAs, and in doing so employing the forward-backward scheme (*Røed*, 2019, Chapter 6) on the Arakawa C-grid displayed by Figure 3), their finite difference versions become

$$h_{jk}^{n+1} = h_{jk}^{n} + \Delta t \left([D_u]_{jk}^{n} + [D_v]_{jk}^{n} \right) \tag{28}$$

$$U_{jk}^{n+1} = U_{jk}^{n} + \Delta t \left([C_u]_{jk}^{n} + [P_u]_{jk}^{n+1} \right), \tag{29}$$

$$V_{jk}^{n+1} = V_{jk}^{n} + \Delta t \left([C_v]_{jk}^{n+1} + [P_v]_{jk}^{n+1} \right), \tag{30}$$

$$\eta_{jk}^{n+1} = h_{jk}^{n+1} - H_0, \tag{31}$$

where

$$[C_u]_{jk}^{n} = \frac{1}{4} f \left(V_{jk-1}^{n} + V_{jk}^{n} + V_{j+1k}^{n} + V_{j+1k-1}^{n} \right), \tag{32}$$

$$[C_v]_{jk}^{n} = -\frac{1}{4} f \left(U_{j-1k}^{n} + U_{j-1k+1}^{n} + U_{jk+1}^{n} + U_{jk}^{n} \right), \tag{33}$$

are the Coriolis terms,

$$[P_u]_{jk}^{n} = -\frac{c_0^2}{\Delta x} \left(\eta_{j+1k}^{n} - \eta_{jk}^{n} \right), \tag{34}$$

$$[P_v]_{jk}^{n} = -\frac{c_0^2}{\Delta y} \left(\eta_{jk+1}^{n} - \eta_{jk}^{n} \right), \tag{35}$$

are the pressure forcing terms, and

$$[D_u]_{jk}^{n} = -\frac{1}{\Delta x} \left(U_{jk}^{n} - U_{j-1k}^{n} \right), \quad [D_v]_{jk}^{n} = -\frac{1}{\Delta y} \left(V_{jk}^{n} - V_{jk-1}^{n} \right) \tag{36}$$

are the divergence terms. Regarding the time levels notice that the pressure term in (29) is evaluated at the new time level, and so is the pressure and Coriolis terms in (30). This is why the scheme is referred to as a forward-backward scheme. The idea behind the FB-L scheme is that as soon as one of the variables are updated to the new time level its value is immediately used in the next equation. This is allowed since the scheme is linear. The FB-L scheme is therefore a so called two time level scheme, which implies that only two time levels have to be stored in memory at any time. The advantage of the two level scheme is that it avoids the initial value problem.

As visualized by (32) and (33) the staggering of the Arakawa C-grid makes the evaluation of the Coriolis terms a bit awkward. Recall that in (29) the Coriolis term requires V in cell (j,k) to evaluated at a U-point, while in (30) it requires U to be evaluated at a

V-point. To arrive at the finite difference equations displayed by (32) and (33) an interpolation using the four nearest points in space is therefore employed. This interpolation may have implications though regarding the implementation of the boundary condition at closed boundaries (reflective walls) as described by *Jamart and Ozer* (1986) and detailed in Section 6.3 below. Finally, it should be emphasized that in order to be numerically stable the FB-L scheme requires that the CFL criteria

$$\Delta t < \frac{\Delta x}{c_0\sqrt{2}}\left[1+\left(\frac{\Delta x}{\Delta y}\right)^2\right]^{-\frac{1}{2}} \tag{37}$$

is satisfied at all times.

5.2 The CTCS-L scheme

Like the FB-L scheme the CTCS-L scheme is a finite difference version of the linear governing equations, that is, (24) through (27), and reads

$$U_{jk}^{n+1} = U_{jk}^{n-1} + 2\Delta t\left([C_u]_{jk}^n + [P_u]_{jk}^n\right), \tag{38}$$

$$V_{jk}^{n+1} = V_{jk}^{n-1} + 2\Delta t\left([C_v]_{jk}^n + [P_v]_{jk}^n\right), \tag{39}$$

$$h_{jk}^{n+1} = h_{jk}^{n-1} + 2\Delta t\left([D_u]_{jk}^n + [D_v]_{jk}^n\right), \tag{40}$$

$$\eta_{jk}^{n+1} = h_{jk}^{n+1} - H_0 \tag{41}$$

where the Coriolis terms $[C_u]_{jk}^n$ and $[C_v]_{jk}^n$, the pressure forcing terms $[P_u]_{jk}^n$ and $[P_v]_{jk}^n$, and the divergence terms $[D_u]_{jk}^n$ and $[D_v]_{jk}^n$ are as given by (32) through (36), respectively. Inspection of the above equations reveals that the CTCS-L scheme is indeed a centered in time, centered in space scheme, and in contrast to the FB-L scheme it is a three level scheme. There are a few important changes to note. Due to the centering in time the factor Δt is replaced by $2\Delta t$, and all terms on the right-hand side of (38) through (40) are evaluated at time level n. It is therefore a true explicit scheme.

In addition, as outlined in *Røed* (2019) (Chapter 6.4), the CFL criterion for stability is slightly more stringent than the criteria given by (37) regarding the FB-L scheme.

Like the FB-L scheme the CTCS-L scheme is afflicted by several disadvantageous properties. Being a three time level scheme the CTCS-L scheme is troubled by the so called initial value problem. In addition it contains numerical or artificial dispersion and has a false computational mode (*Røed*, 2019).

Regarding the artificial dispersion the numerical dispersion relation for the CTCS-L scheme in one dimension (*Røed*, 2019, Chapter 6.4, page 130) reads

$$c_{1,2}(\alpha) = \pm \frac{1}{\alpha \Delta t} \arcsin \left[c_0 \frac{\Delta t}{\Delta x} \sin(\alpha \Delta x) \sqrt{1 + \left(\frac{\Delta x}{L_r \sin(\alpha \Delta x)} \right)^2} \right], \tag{42}$$

where α is the wavenumber, $L_r = c_0/f$ is Rossby's deforamtion radius, and $c_{1,2}$ is the numerical phase speed. The numerical phase speed is therefore a function of the wavelength. If $\alpha \Delta x$ is large that particular wavelength is not well resolved, and the numerical phase speed becomes significantly smaller than the true phase speed. Consequently, the classic sign of numerical dispersion is a trail of waves of various wavelengths, usually shorter than $4\Delta x$, lagging behind the main distribution.

The initial problem is common to all centered in time schemes (*Røed*, 2019, page 84, Chapter 5.6). It arises since knowledge of the variables at time t^{-1} is required when solving for the first time level $n = 1$. Since the governing equations only allow specification of three initial conditions, the specification of the variables at both $t = t^0$ and $t = t^{-1}$ is not allowed. The common remedy is to switch to a forward in time, centered in space scheme (FTCS) to perform the first integration step in time. The simplest FTCS scheme to employ is the Euler scheme. To arrive at at an Euler scheme the superscipts $n-1$ appearing on the right-hand side of (38), (39) and (40) are replaced by n and the factor $2\Delta t$ on the right-hand side is replaced by Δt. By doing so the Euler scheme reads

$$U_{jk}^{n+1} = U_{jk}^{n} + \Delta t \left([C_u]_{jk}^{n} + [P_u]_{jk}^{n} \right), \tag{43}$$

$$V_{jk}^{n+1} = V_{jk}^{n} + \Delta t \left([C_v]_{jk}^{n} + [P_v]_{jk}^{n} \right), \tag{44}$$

$$h_{jk}^{n+1} = h_{jk}^{n} + \Delta t \left([D_u]_{jk}^{n} + [D_v]_{jk}^{n} \right), \tag{45}$$

$$\eta_{jk}^{n+1} = h_{jk}^{n+1} - H_0. \tag{46}$$

Finally keep in mind that the Euler scheme is unconditionally unstable. Nevertheless to employ the Euler scheme once does not ruin the stability. Moreover, it may be done from time to time to rule out the computational mode inherent in all CTCS schemes (cf. *Røed*, 2019, Chapter 5.8).

5.3 The CTCS-N scheme

To solve the fully, nonlinear governing equation (19) through (22) a second order, centered in time centered in space scheme is employed. Due to the inclusion of the nonlinear terms

the CTCS-N scheme is susceptible to nonlinear instabilities. An artificial eddy viscosity term is therefore added to suppress these instabilities if needed (cf. *Røed*, 2019, Chapter 10.3).

Again the same Arakawa C-grid (Figure 3) is employed. The finite difference version of (19) through (22) then take the forms

$$U_{jk}^{n+1} = \left\{U_{jk}^{n-1} + 2\Delta t\left([C_u]_{jk}^n + [P_u]_{jk}^n + [A_u^x]_{jk}^n + [A_u^y]_{jk}^n + [M_u]_{jk}^n\right)\right\}R, \tag{47}$$

$$V_{jk}^{n+1} = \left\{V_{jk}^{n-1} + 2\Delta t\left([C_v]_{jk}^n + [P_v]_{jk}^n + [A_v^x]_{jk}^n + [A_v^y]_{jk}^n + [M_v]_{jk}^n\right)\right\}R, \tag{48}$$

$$h_{jk}^{n+1} = h_{jk}^{n-1} + 2\Delta t\left([D_u]_{jk}^n + [D_v]_{jk}^n\right), \tag{49}$$

$$\eta_{jk}^{n+1} = h_{jk}^{n+1} - H_{jk} \tag{50}$$

where the factor

$$R = \frac{1}{1 + 2A\Delta t\left(\frac{1}{\Delta x^2} + \frac{1}{\Delta y^2}\right)} \tag{51}$$

appears due to the application of the Dufort-Frankel FDA of the eddy viscosity (mixing) terms. Since the Coriolis terms $[C_u]_{jk}^n$ and $[C_v]_{jk}^n$ and the divergence terms $[D_u]_{jk}^n$ and $[D_v]_{jk}^n$ are linear terms they are as listed by (32), (33) and (36), respectively. The pressure terms, however, are nonlinear and now take the forms

$$[P_u]_{jk}^n = -\frac{g}{2\Delta x}\left(h_{j+1k}^n + h_{jk}^n\right)\left(\eta_{j+1k}^n - \eta_{jk}^n\right), \tag{52}$$

$$[P_v]_{jk}^n = -\frac{g}{2\Delta y}\left(h_{jk+1}^n + h_{jk}^n\right)\left(\eta_{jk+1}^n - \eta_{jk}^n\right). \tag{53}$$

In addition the nonlinear momentum flux terms and mixing terms are added. The former take the finite difference forms

$$[A_u^x]_{jk}^n = -\frac{1}{4\Delta x}\left[\frac{\left(U_{j+1k}^n + U_{jk}^n\right)^2}{h_{j+1k}^n} - \frac{\left(U_{jk}^n + U_{j-1k}^n\right)^2}{h_{jk}^n}\right], \tag{54}$$

$$[A_u^y]_{jk}^n = -\frac{1}{\Delta y}\left[\frac{\left(U_{jk+1}^n + U_{jk}^n\right)\left(V_{j+1k}^n + V_{jk}^n\right)}{h_{jk}^n + h_{jk+1}^n + h_{j+1k+1}^n + h_{j+1k}^n} - \frac{\left(U_{jk}^n + U_{jk-1}^n\right)\left(V_{j+1k-1}^n + V_{jk-1}^n\right)}{h_{jk-1}^n + h_{jk}^n + h_{j+1k}^n + h_{j+1k-1}^n}\right]. \tag{55}$$

$$[A_v^x]_{jk}^n = -\frac{1}{\Delta x}\left[\frac{\left(U_{jk+1}^n + U_{jk}^n\right)\left(V_{j+1k}^n + V_{jk}^n\right)}{h_{jk}^n + h_{jk+1}^n + h_{j+1k+1}^n + h_{j+1k}^n} - \frac{\left(U_{j-1k+1}^n + U_{j-1k}^n\right)\left(V_{jk}^n + V_{j-1k}^n\right)}{h_{j-1k}^n + h_{j-1k+1}^n + h_{jk+1}^n + h_{jk}^n}\right], \tag{56}$$

and

$$[A_v^y]_{jk}^n = -\frac{1}{4\Delta y}\left[\frac{\left(V_{jk+1}^n + V_{jk}^n\right)^2}{h_{jk+1}^n} - \frac{\left(V_{jk}^n + V_{jk-1}^n\right)^2}{h_{jk}^n}\right], \tag{57}$$

respectively, while the mixing terms take the forms

$$[M_u]_{jk}^n = \frac{A}{\Delta x^2}\left(U_{j+1k}^n - U_{jk}^{n-1} + U_{j-1k}^n\right) + \frac{A}{\Delta y^2}\left(U_{jk+1}^n - U_{jk}^{n-1} + U_{jk-1}^n\right), \tag{58}$$

and

$$[M_v]_{jk}^n = \frac{A}{\Delta x^2}\left(V_{j+1k}^n - V_{jk}^{n-1} + V_{j-1k}^n\right) + \frac{A}{\Delta y^2}\left(V_{jk+1}^n - V_{jk}^{n-1} + V_{jk-1}^n\right). \tag{59}$$

Just like the CTCS-L scheme an Euler scheme has to be employed to start the integration. Thus, the superscipt $n-1$ appearing in the first terms on the right-hand sides of (47), (48) and (49) is replaced by n, and the factor $2\Delta t$ by Δt. In addition the Dufort-Frankel FDA used for the mixing terms is abandoned and replaced by an ordinary second order FDA. By doing so the Euler scheme takes the form

$$\begin{aligned}
U_{jk}^{n+1} &= U_{jk}^n + \Delta t\left([C_u]_{jk}^n + [P_u]_{jk}^n + [A_u^x]_{jk}^n + [A_u^y]_{jk}^n + [M_u^*]_{jk}^n\right), && (60)\\
V_{jk}^{n+1} &= V_{jk}^n + \Delta t\left([C_v]_{jk}^n + [P_v]_{jk}^n + [A_v^x]_{jk}^n + [A_v^y]_{jk}^n + [M_v^*]_{jk}^n\right), && (61)\\
h_{jk}^{n+1} &= h_{jk}^n + \Delta t\left([D_u]_{jk}^n + [D_v]_{jk}^n\right), && (62)\\
\eta_{jk}^{n+1} &= h_{jk}^{n+1} - H_{jk}. && (63)
\end{aligned}$$

where

$$[M_u^*]_{jk}^n = \frac{A}{\Delta x^2}\left(U_{j+1k}^n - 2U_{jk}^n + U_{j-1k}^n\right) + \frac{A}{\Delta y^2}\left(U_{jk+1}^n - 2U_{jk}^n + U_{jk-1}^n\right), \tag{64}$$

and

$$[M_u^*]_{jk}^n = \frac{A}{\Delta x^2}\left(V_{j+1k}^n - 2V_{jk}^n + V_{j-1k}^n\right) + \frac{A}{\Delta y^2}\left(V_{jk+1}^n - 2V_{jk}^n + V_{jk-1}^n\right). \tag{65}$$

It is emphasized that interpolation is used to arrive at the various FDAs listed above. Thus some of the interpolations must be rewritten close to solid, reflective walls. In particular this is true regarding the Coriolis terms (cf. Section 6.3 below).

6 Boundary conditions

6.1 Conditions at the northern and southern walls

Along the southern and northern boundaries the no-slip condition (11) and (12) requires U as well as V to be zero there. Since the auxiliary points are on the boundary the FDA of (11) and (12) read

$$U_{j1}^{+n} = V_{j1}^{n} = 0, \quad \text{and} \quad U_{jK}^{+n} = V_{jK+1}^{n} = 0 \quad ; \quad j = 2(1)J-1, \quad \forall n, \tag{66}$$

respectively. As before the notation U_{j1}^{+n} and U_{jK}^{+n} entails evaluation of U at an auxiliary point in the cells $(j,1)$ and (j,K), respectively. Performing the interpolation $U_{jk}^{+n} = \frac{1}{2}\left(U_{jk}^{n} + U_{jk+1}^{n}\right)$ (66) takes the final form

$$U_{j1}^{n} = -U_{j2}^{n}, \quad \text{and} \quad V_{j1}^{n} = 0 \quad ; \quad j = 2(1)J-1, \quad \forall n, \tag{67}$$

and

$$U_{jK+1}^{n} = -U_{jK}^{n}, \quad \text{and} \quad V_{jK+1}^{n} = 0 \quad ; \quad j = 2(1)J-1, \quad \forall n. \tag{68}$$

The closed boundary condition is therefore fulfilled by letting the values of the U-points just outside the wall being mirrored across the model ocean boundary along the northern and western boundaries. This explains why ghost cells have to be added outside of the northern boundary.

6.2 Conditions at the open western and eastern boundaries

Along the eastern and western boundaries either the cyclic boundary condition or the FRS is imposed. At open boundaries the governing equations are still valid. Thus, first U, V and h are computed at the eastern boundary $j = J+1$ using the schemes above. While doing this any references to $j = J+2$ must be replaced by $j = 2$ to satisfy the cyclic boundary condition. Next, the cyclic boundary condition (13) requires that the solutions at the western boundary $j = 1$ are updated according to

$$U_{1k}^{n} = U_{J+1k}^{n}, \quad V_{1k}^{n} = V_{J+1k}^{n} \quad \text{and} \quad h_{1k}^{n} = h_{J+1k}^{n} \tag{69}$$

for $k = 1(1)K+1$ and for all n.

When using the FRS as the open boundary condition the N_g points closest to western and eastern boundaries are first set aside to cover so called FRS zones, the remaining points being referred to as interior points (cf. *Røed*, 2019, Figure 7.4, on page 172). Here N_g must be a number larger than or equal to 7 (*Engedahl*, 1995a). Within the FRS zones the solutions are relaxed towards specified solutions at the western and eastern boundaries. The resulting solutions are therefore only valid within the interior domain. The FRS works

as a predictor-corrector scheme. In the first step, the predictor step, the integration of the numerical scheme is performed as usual, but excludes the western and eastern boundary points (the cells $j = 1$ and $j = J+1$). Letting the variables be represented by the vector $\mathbf{Y} = [U,V,h,\eta]^T$ the first step is performed by computing it from the numerical scheme in question. For instance regarding the FB-L scheme it would read

$$\mathbf{Y}^*_{jk} = \mathbf{Y}^n_{jk} + \mathscr{L}[\mathbf{Y}^n_{jk}]; \quad j = 2(1)J, k = 2(1)K. \tag{70}$$

while for the CTCS-L and CTS-N it would read

$$\mathbf{Y}^*_{jk} = \mathbf{Y}^{n-1}_{jk} + \mathscr{L}[\mathbf{Y}^n_{jk}]; \quad j = 2(1)J, k = 2(1)K. \tag{71}$$

where $\mathscr{L}$ is an operator operating on $\mathbf{Y}^n_{jk}$ in accord with the chosen scheme and $\mathbf{Y}^*_{jk}$ is the predictor. The next step, the corrector step, is performed by correcting the predictor solution by relaxing it towards an exterior solution, that is,

$$\mathbf{Y}^{n+1}_{jk} = (1-\alpha_j)\mathbf{Y}^*_{jk} + \alpha_j \mathbf{Y}^e_{jk}; \quad j = 2(1)J, k = 2(1)K, \tag{72}$$

$$\mathbf{Y}^{n+1}_{1k} = \mathbf{Y}^e_{1k} \quad \text{and} \quad \mathbf{Y}^{n+1}_{J+1k} = \mathbf{Y}^e_{J+1k}; \quad k = 2(1)K \tag{73}$$

Here $\mathbf{Y}^e_{jk}$ is the external solution, while α_j is a relaxation parameter defined by

$$\alpha_j = \begin{cases} 1 - \tanh\left(\frac{j-1}{3}\right) & \text{if } 1 < j < N_g, \\ 0 & \text{if } N_g \le j \le J+1-N_g, \\ 1 - \tanh\left(\frac{J+1-j}{3}\right) & \text{if } J+1-N_g < j < J+1. \end{cases} \tag{74}$$

It should be emphasized that this definition requires that N_g is sufficiently large so that α is approaching one at the exterior end of the FRS zones, e.g., for $j = 1$, and that its gradient is close to zero at the interior end of the FRS zones, e.g., $j = N_g$.

6.3 The Coriolis terms near closed boundaries

Since all schemes employ the same staggered C-grid, the Coriolis terms are the same for all three schemes. Near closed boundaries, such as the northern and southern boundaries, the interpolation due to the staggered C-grid uses values at points where $V^n_{jk} = 0$ due to the boundary condition. For instance close to the southern boundary at $k = 2$ (32) reads

$$[C_u]^n_{j2} = \frac{1}{4} f\left(V^n_{j1} + V^n_{j2} + V^n_{j+12} + V^n_{j+11}\right); \quad j = 2(1)J. \tag{75}$$

Here V^n_{j1} and V^n_{j+11} are located at the southern boundary, and hence vanishes in accord with (66). If the factor $\frac{1}{4}$ in front of (75) is kept then, as shown by *Jamart and Ozer* (1986), artificial boundary currents will appear. By replacing the factor $\frac{1}{4}$ in (75) with $\frac{1}{2}$ they showed that the boundary currents disappeared, and concluded that they were a true numerical artifact. The same is true when considering the points closest to the northern boundary ($k = K$).

Consequently the Coriolis term in (32) is multiplied by a function defined by

$$\delta_k = f \begin{cases} \frac{1}{4} & \text{if } k = 3(1)K-1 \\ \frac{1}{2} & \text{if } k = 2 \text{ or } k = K \end{cases}. \tag{76}$$

to read

$$[C_u]^n_{jk} = \delta_k \left(V^n_{jk-1} + V^n_{jk} + V^n_{j+1k} + V^n_{j+1k-1} \right) \; ; \; j = 2(1)J, \, k = 2(1)K. \tag{77}$$

7 Cases studied

As alluded to in the introduction (Section 1) four cases are considered. They are identical to cases presented by *Holm et al.* (2020), which were constructed to test the ability of the schemes presented in Section 5 to replicate a shock (Case I), adjustment under gravity (Case II), and the advection of small and large amplitude Kelvin waves (Cases III and IV). The salient numbers such as length and width of the model ocean domain, mesh sizes, etc, are listed in Table 1. Details regarding the respective cases are given by Sections 7.1 through 7.4 below. Note that in accord with (15)

$$L_x = J\Delta x, \quad \text{and} \quad L_y = (K-1)\Delta y, \tag{78}$$

where L_x, L_y are the physical dimensions of the model ocean domain in the east-west and north-south direction, respectively (Figure 2). Thus $L_x, J, \Delta x$ are interdependent and so are $L_y, K, \Delta y$. Hence given two of them the third must be computed from (78).

7.1 Case I: Dam break on a wet domain

Corresponds to Case A of *Holm et al.* (2020). It has no rotation and is therefore a one-dimensional problem. It is the traditional and commonly used test case of non-rotating fluid dynamics to test a scheme's ability to represent shocks. It is initiated as a state of rest ($U = V = 0$) at time $t = 0$ ($n = 0$), but features an initial (small) step in the sea

Table 1: Symbols and values of parameters used to initialize the variables and to run the four test cases. L_x, L_y are the dimensions of the model ocean domain in the east-west and north-south direction, respectively (Figure 2). H_0 is the equilibrium depth of the model ocean, f is the Coriolis parameter, g is the gravitational acceleration, η_0 is the amplitude, or maximum, of the initial sea surface deviation, and x_0 and y_0 correspond to its location in the Cartesian coordinate system (Figure 3). Furthermore, $J+1, K+1$ are the number of cells in the x, y directions, respectively, while $\Delta x, \Delta y$ are the space increments in those directions, Δt is the time step, and A is the eddy viscosity coeffecient for the CTCS-N scheme only. Finally, N_{fl} denotes the number of time steps necessary for a linear Kelvin wave to propagate across the model domain in the east-west direction once, and hence is applicable to Cases III and IV only.

Symbol	Case I	Case II	Case III	Case IV	Unit
L_x	10^{-2}	40000	5000	5000	km
L_y	10^{-3}	50000	2000	2000	km
H_0	0.005	1000	100	100	m
f	0	$1.2 \cdot 10^{-4}$	$1.2 \cdot 10^{-4}$	$1.2 \cdot 10^{-4}$	s^{-1}
g	9.81	9.81	9.81	9.81	ms^{-2}
η_0	0.002	0.2	0.05	2.0	m
x_0	$5 \cdot 10^{-3}$	20000	2500	2500	km
y_0	-	25000	-5	-5	km
$J+1, K+1$	$500/51$	$800, 1001$	$1000, 201$	$1000, 201$	-
$\Delta x, \Delta y$	$2 \cdot 10^{-5}$	50	5,10	5,10	km
Δt	0.01	100	$\frac{L_x}{N_{fl}\sqrt{gH_0}} \approx 31.93$	$\frac{L_x}{N_{fl}\sqrt{gH_0}} \approx 31.93$	s
A	0.1	25	25	25	m^2s^{-1}
N_{fl}	-	-	5000	5000	-

surface deviation in middle of the domain which is then released (Figure 4). At the open boundaries at the western and eastern end of the domain the FRS is imposed as the open boundary condition (Section 6).

The initial sea surface deviation is such that it has a positive step westward with a

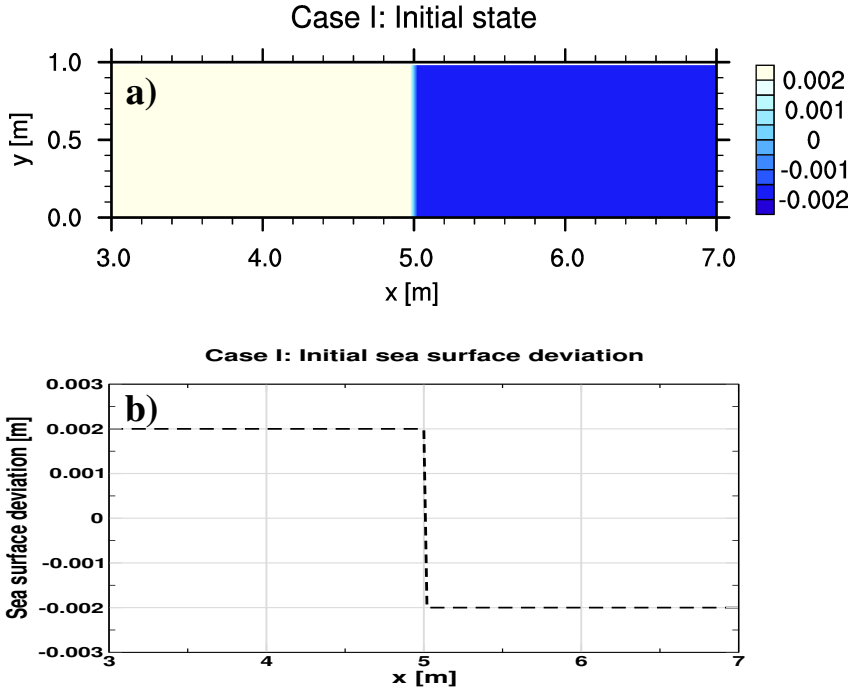

Figure 4: Case I: The initial sea surface deviation η in meters. Distances are in meters. Distance in the east-west direction is truncated. a) Top view, and b) side view.

similar negative step eastward. Thus,

$$\eta(x,y,0) = \eta_0 \begin{cases} +1 & \text{if } 0 < x \leq x_0, \\ 0 & \text{if } x = x_0, \\ -1 & \text{if } x_0 < x \leq L_x, \end{cases} \quad \text{and} \quad U(x,y,0) = V(x,y,0) = 0. \tag{79}$$

Here η_0 is the the amplitude of the step and $x_0 = L_x/2$ denotes the location halfway between the western and eastern boundaries (Table 1).

The FDA of (79) is

$$\eta^0_{jk} = \eta_0 \begin{cases} +1 & \text{if } 0 \leq j \leq j_0, \\ -1 & \text{if } j_0 < j \leq J+1, \end{cases} \quad \text{and} \quad U^0_{jk} = V^0_{jk} = 0. \tag{80}$$

where j_0 is the cell number where the location of the auxiliary point equals x_0. Recall that origo is at the auxilary point in cell (1,1) (Figure 3). Thus, $j_0 = 1 + \frac{L_x}{2\Delta x}$, where Δx is the space increment and L_x the length of the model ocean domain in the east-west direction (Table 1 for actual values). The initial condition for the sea surface deviation is therefore as displayed by Figure 4.

7.2 Case II: Rossby adjustment

Corresponds to Case B of *Holm et al.* (2020). It is performed on a rotating earth. The initial state is unbalanced so that $U = V = 0$ (zero transport), while the sea surface has a smooth hump in the middle of the domain (Figure 5). This is the traditional test case used in geophysical fluid dynamics to test a scheme's ability to replicate the steady state solution of geostrophic balance as time goes to infinity. Analytic solutions to this problem with an initial step function was already derived by C. G. Rossby in the 1930s (*Rossby*, 1937, 1938). The FRS is imposed as the open boundary condition at the western and eastern boundaries.

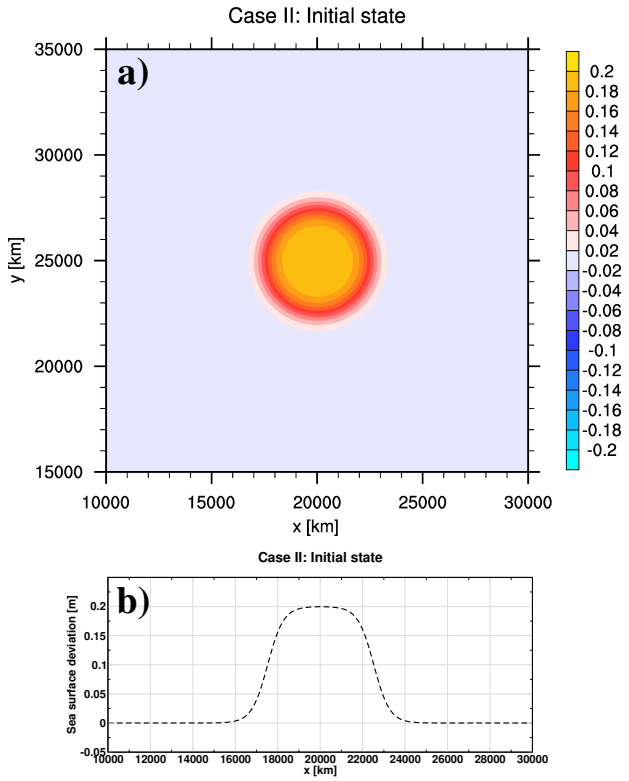

Figure 5: Case II: The initial sea surface deviation in meters. a) Top view. Numbers along axes are distance (unit is km, truncated in both direction). b) Side view is a cut along $y = y_0$. Numbers along the horizontal axis are distance (unit km) in the eastern direction (also truncated). The vertical axis indicates sea surface deviation (unit m).

The sea surface deviation is given in accord with the formula of *Holm et al.* (2020)

(their Section 4.2), that is,

$$\eta(x,y,0) = \frac{1}{2}\eta_0 \left[1 + \tanh\left(\frac{-\sqrt{(x-x_0)^2+(y-y_0)^2}+D}{L}\right)\right]. \tag{81}$$

Here $D = 50\Delta x$ relates to the width of the hump, while $L = 15\Delta x$ relates to its steepness. Other values are give by Table 1.

With reference to Figure 3 the numerical rendition of the initial conditions take the forms

$$U^0_{jk} = V^0_{jk} = 0, \tag{82}$$

and

$$\eta^0_{jk} = \frac{1}{2}\eta_0 \left[1 + \tanh\left(\frac{-\sqrt{(x_j - 0.5\Delta x - x_0)^2+(y_k - 0.5\Delta y - y_0)^2}+D}{L}\right)\right], \tag{83}$$

for all $j = 1(1)J$ and $k = 1(1)K$, respectively. The initial two-dimensional hump is visualized by Figure 5.

7.3 Case III: Small Kelvin waves

Corresponds to Case D of *Holm et al.* (2020), and is also very similar to Case IV (Section 7.4). In both cases the initial state consists of a wave trapped at the southern boundary and travelling eastwards. The wave has its maximum at the southern boundary and decays exponentially towards the northern boundary (Figure 6). It is a balanced state so that the initial pressure gradient caused by the specified sea surface deviation is balanced by the Coriolis force. Consequently $U \neq 0$ while $V = 0$.

Case III and Case IV test a scheme's ability to properly represent Kelvin waves, which are ubiqutous in the ocean. The only difference between the two cases is the initial maximum amplitude of the Kelvin wave. In Case III it is $\eta_0 = 0.05$ m, while in Case IV it is raised by a factor of 40 to $\eta_0 = 2$ m. Accordingly, the response regarding Case III is expected to be linear, while the response for Case IV is expected to be highly nonlinear. Both cases make use of a cyclic boundary condition at the eastern and western boundaries.

In accord with *Holm et al.* (2020) the initial sea surface deviation takes the form

$$\eta(x,y,0) = \frac{1}{2}\eta_0 \exp\left(\frac{-\sqrt{(y-y_0)^2}}{L_r}\right)\left[1 + \tanh\left(\frac{-\sqrt{(x-x_0)^2}+L_r}{L_r/3}\right)\right], \tag{84}$$

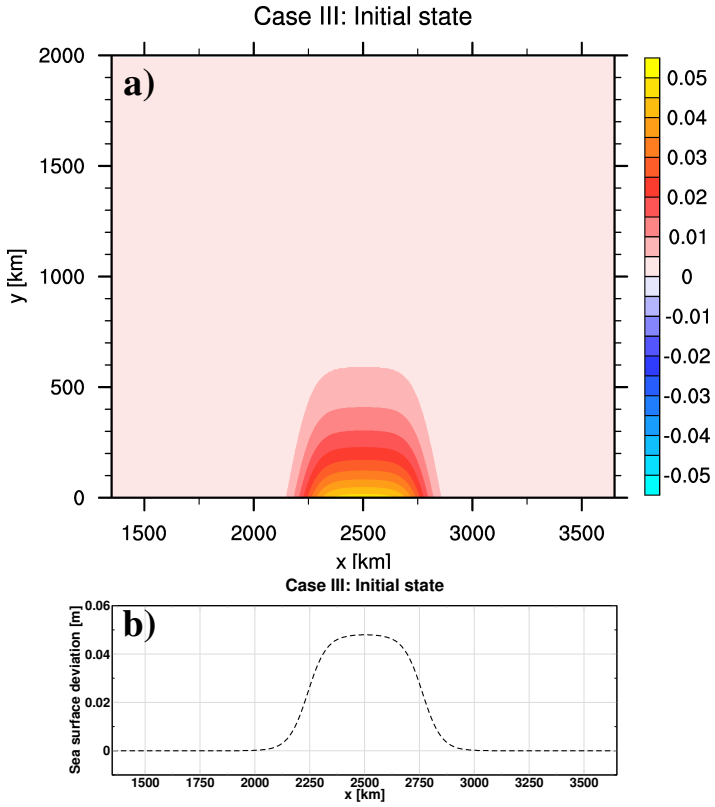

Figure 6: Case III: The initial sea surface deviation in meters. a) Top view. Numbers along axes are distance (unit km) in the eastern (horizontal) and northern (vertical) direction, respectively. The distance in the eastern direction is truncated. b) Side view. The cut is along $y = 0.5\Delta y$. Numbers along horizontal axis as in Panel a. Sea surface deviation (unit m) is indicated along the vertical axis.

while the transport in the eastern direction is

$$U(x,y,0) = c_0 \,\mathrm{sgn}\,(y-y_0)\,\eta(x,y,0) \quad \text{and} \quad V(x,y,0) = 0. \tag{85}$$

In (84) and (85) $c_0 = \sqrt{gH_0}$ is the phase speed of linear Kelvin waves and $L_r = c_0/f$ is Rossby's deformation radius. Details regarding the other parameters are given by Table 1.

The numerical rendition of (84) is simply

$$\eta^0_{jk} = \frac{1}{2}\eta_0 \exp\left(\frac{-\sqrt{(y_k - 0.5\Delta y - y_0)^2}}{L_r}\right)\left[1 + \tanh\left(\frac{-\sqrt{(x_j - 0.5\Delta x - x_0)^2} + L_r}{L_r/3}\right)\right]. \tag{86}$$

for all $j = 1(1)J$ and $k = 1(1)K$. Recalling that U-points are staggered one half space

increment relative to the η-points the numerical rendition of (85) is

$$U^0_{jk} = \frac{1}{2} c_0 \,\mathrm{sgn}\,(y_k - 0.5\Delta y - y_0)\left(\eta^0_{j+1k} + \eta^0_{jk}\right) \quad \text{and} \quad V^0_{jk} = 0, \tag{87}$$

again for all $j = 1(1)J$ and $k = 1(1)K$.

7.4 Case IV: Large Kelvin waves

As shown by Figure 7 Case IV is similar to Case III, the only difference being that the initial maximum amplitude of the hump is raised 40 times to $\eta_0 = 2$ m (Table 1).

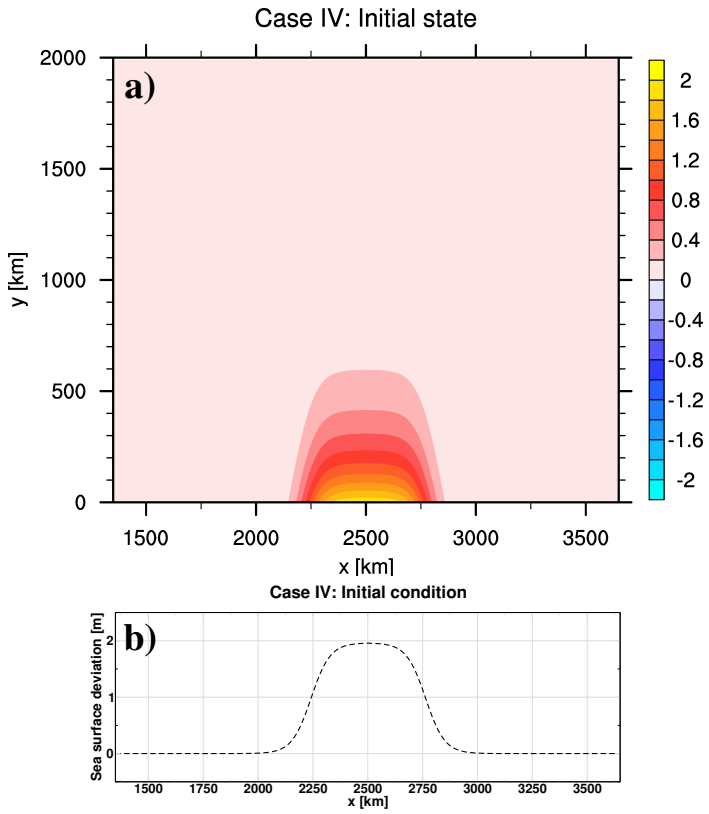

Figure 7: Case IV: As Figure 6, but with the initial sea surface deviation amplitude raised to 2 m (a factor of 40).

Due to the extremely high initial water level it is expected that the response will be highly nonlinear in that the nonlinear terms, e.g., the advective fluxes, will be of importance. It is therefore expected, as already shown by *Holm et al.* (2020), that the solution provided by schemes that contain the nonlinear terms, e.g., CTCS-N, will be dramatically different from the solutions provided by linear schemes such as FB-L and CTCS-L. On the other hand the two latter schemes should be very similar to each other, and also to the linear solutions presented by *Holm et al.* (2020).

8 Results and discussions

As mentioned in the introductory section (Section 1) and in Section 7 two of the schemes used in the present study were also among the four schemes tested by *Holm et al.* (2020). Specifically, the schemes they called FBL and CTCS correspond exactly to the finite difference schemes FB-L and CTCS-N described above (Section 5). In addition the four cases studied here were also part of their investigation. A comparison with their results and those presented here is therefore possible. *Holm et al.* (2020) also included two additional schemes, namely the KP scheme (credited to *Kurganov and Petrova*, 2007) and the CDKLM scheme (credited to *Chertok et al.*, 2018). They are both recent well-balanced, finite volume schemes constructed among other things to handle shocks. It is therefore of interest to compare the solutions derived here with their solutions derived using the KP and CDKLM schemes.

Finally, recall that *Holm et al.* (2020) utilized the computer's Graphical Processing Unit (GPU). Hence they programmed and compiled all the schemes using a GPU programming language. This is contrast to the present study where the programming language utilized is FORTRAN 95, and which was compiled and run on the computer's Central Processing Unit (CPU).

8.1 Case I: Dam break on a wet domain

The solutions obtained using the FB-L, CTCS-L and CTCS-N schemes after 6 seconds of integration are displayed by Figure 8 together with the analytic solution (*Delestre et al.*, 2013, Section 4.1.1, page 23). As expected it is impossible to distinguish between the two linear solutions derived using the schemes FB-L and CTCS-L. In contrast the solution provided using the nonlinear scheme CTCS-N is quite different, and more in line with the analytic solution.

The most prominent feature of all three solutions are the trailing waves. As outlined in Section 5 this is hardly surprising since all three schemes contain numerical dispersion implying that the numerical phase speed is a function of the wave number. Consequently the slower waves lags behind. As a result all the three schemes show trailing waves behind the moving shock, which is the classic sign of numerical dispersion.

Because of its nonlinearity the CTCS-N scheme differs from the two linear schemes which have oscillations on both fronts in line with (42). In contrast the nonlinear scheme

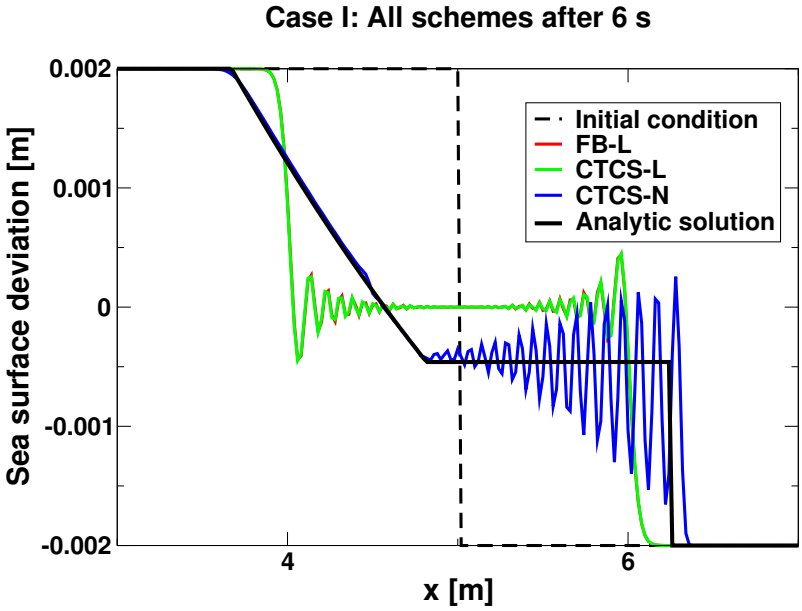

Figure 8: Case I: The sea surface deviation η as a function of along channel distance.

have almost no oscillations on the westward moving front. Moreover the sea surface deviation is lowered on the right-hand side of the initial step compared with the two linear solutions, and the oscillations have a larger amplitude. Note also that the fronts on both sides of the initial step has moved ahead of the linear solutions which moves at the linear phase speed $c_0 = \sqrt{gH_0}$. All of these features, including the dispersion, are in line with what to expect when using a nonlinear finite difference scheme.

As shown by Figure 8 all schemes are off target compared with the analytic solution. It is satisfying though to observe that the two linear solutions, which are indistinguisable from each other, are quite similar to the linear solution presented by *Holm et al.* (2020) (the FBL scheme of their Figure 6). In contrast the present solution derived using the nonlinear CTCS-N scheme disagrees strongly with the nonlinear CTCS solution presented by *Holm et al.* (2020). In fact, except for the numerical dispersion causing the trailing waves lagging behind the right-hand moving front, it appears to be more in line with the analytic solution, and interestingly also more in line with the KP and CDKLM solutions of *Holm et al.* (2020) (their Figure 6).

Since the CTCS-N scheme may be nonlinearly unstable it is of interest to investigate whether the dispersive waves grow in time and eventually causes the solution to go nonlinearly unstable. Consequently an additional longer run was performed with no eddy viscosity applied ($A = 0$ m^{2}s^{-1}). The results is shown by Figure 9. The integration was

stopped after 22 seconds just before the front reached the end of the FRS zones on both sides. As revealed the waves appears immedately after the integration is started, and do grow in amplitude albeit very slowly. Thus within the integration time the nonlinear instability does not seem to be a problem. However, a second additional run for 24 s (not shown) did break down, but its breakdown is attributed to the FRS's inability to handle the waves and not to nonlinear instability. The latter was confirmed by performing yet a third additional run with a domain twice as big for 30 seconds in which the resolution was kept the same (not shown).

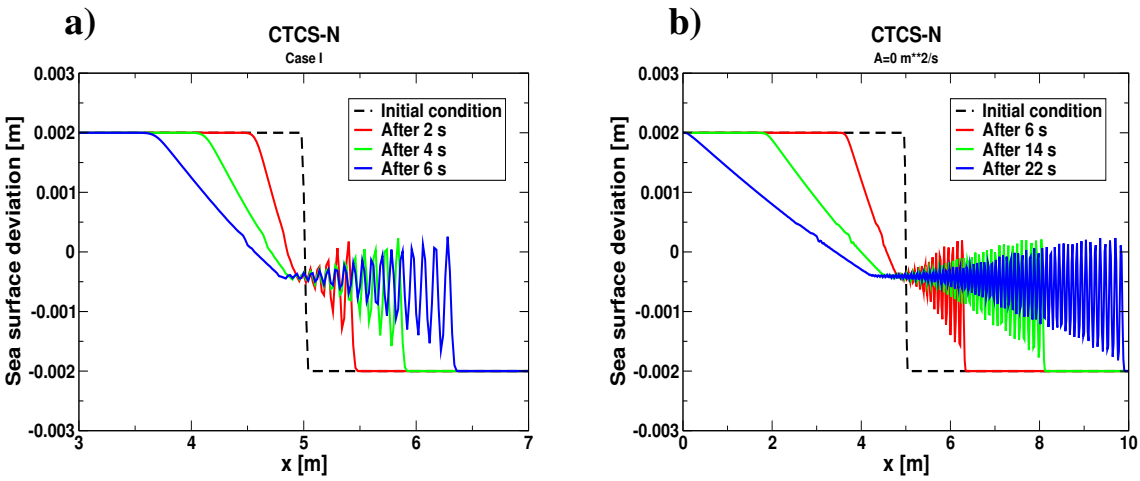

Figure 9: Results applying the CTCS-N scheme. a) Showing the solution after 2, 4 and 6 seconds for a truncated domain. b) Showing the results after 6, 14 and 22 seconds for the whole domain. The solution after 22 seconds is just before the fronts hit the end of the FRS zones on both sides of the computational domain.

To conclude all three schemes behave as expected, but all deviate from the true solution due to their inability to handle the shock. Of the three solutions the solution provided using the CTCS-N scheme is the only one proffering something similar to the true solution. Moreover, the latter solution deviates radically from the CTCS solution of *Holm et al.* (2020) (their Figure 6), a solution very dissimilar to the true solution. It is therefore reasonable to conclude that there is a bug regarding the implementation of the their nonlinear CTCS scheme (most probably a programming error). This conclusion is further corroborated by the Case IV results (Section 8.4), where the solution provided using the present CTCS-N scheme again differs fundamentally from the solution provided by the CTCS scheme of *Holm et al.* (2020).

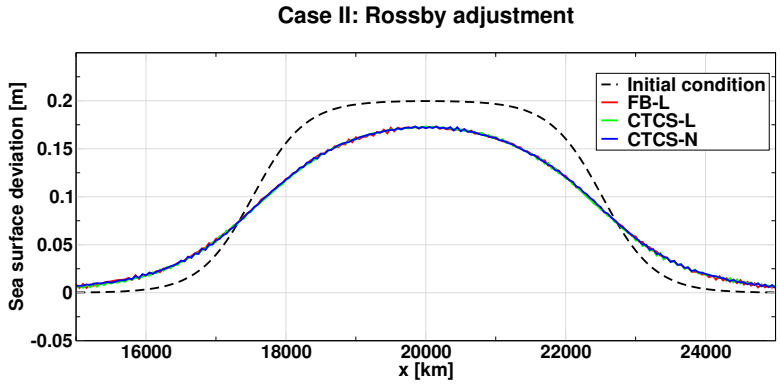

Figure 10: Case II: The solutions in terms of the sea surface deviation as a function of distance in the east-west direction at $y = y_0$ km after approximately 150 days. The east-west distance has been truncated.

8.2 Cases II: Rossby adjustment

The solutions derived in terms of the sea surface deviation after approximately 150 days of integration are displayed by Figure 10. At this stage all three solutions have reached a steady state. It is therefore gratifying to notice that the three schemes provide almost indistinguishable solutions, even the nonlinear CTCS-N scheme. This is hardly surprising though, since the steady state solution is a solution to the linear Klein-Gordon equation (cf. *Holm et al.*, 2020, Section A.5, eq. A.37) as a result of potential vorticity conservation (cf. *Røed*, 2019, Section 6.3, page 125).

Another important feature is that the steady state solution contains both potential and kinetic energy. This is an effect of including rotation. Without rotation the steady state would have been one at rest in which all the initial (available) potential energy would have been lost. Due to rotation a state in which the pressure gradient is balanced by the Coriolis force, the so called geostrophic balance (e.g., *Røed*, 2019, Section 1.6, page 9), is reached instead. The steady state therefore contains both potential as well as kinetic energy. As visualized by Figures 10 and 11 the bump is hence still present although somewhat reduced in amplitude and somewhat wider than it was initially. To balance the pressure gradient a motion exists in the steady state as displayed by Figure 12.

Since the initial state is unbalanced a motion ensues as soon as the integration starts by releasing potential energy. To begin with the released potential energy is partly converted into inertia-gravity wave energy and partly into kinetic energy (*Rossby*, 1937). Although the inertia gravity waves are still present after 3 days of integration, as illustrated by Figure

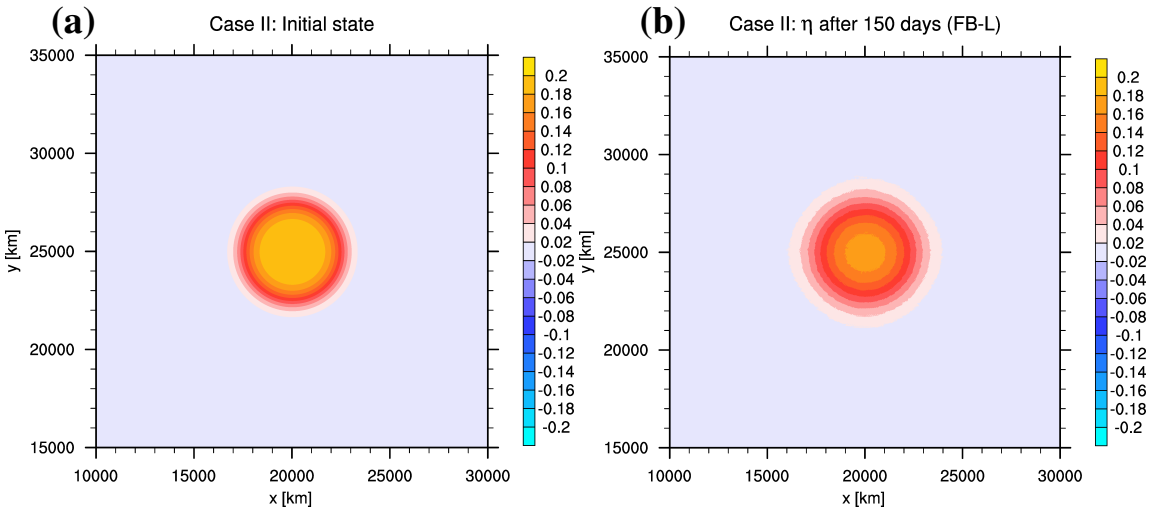

Figure 11: Case II: Panel a) shows the initial state and panel b) the state after 150 days. Numbers along the axes are distance in km. Note that zeroes are not plotted.

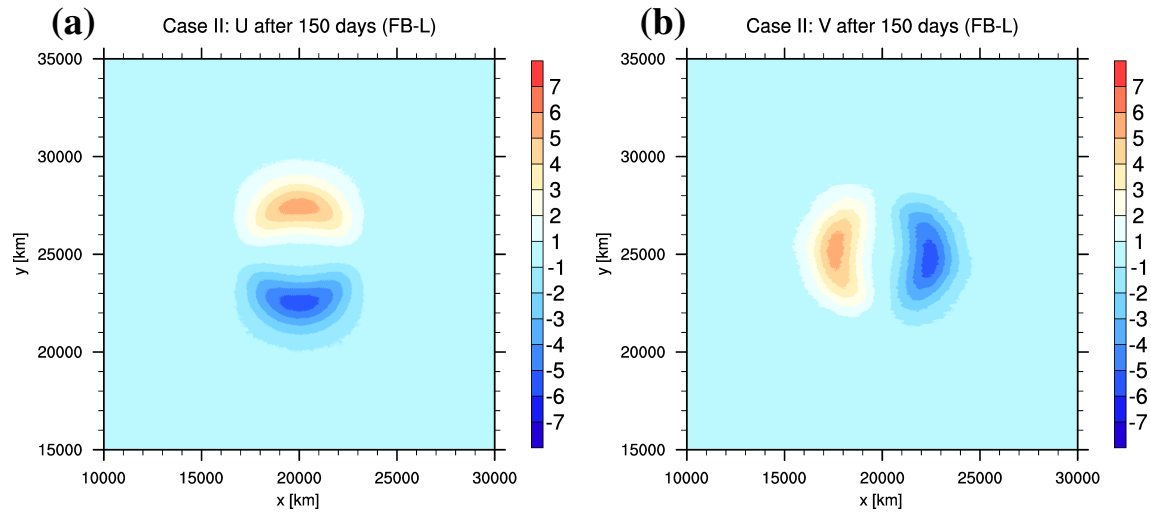

Figure 12: As Figure 11, but showing a) the north-south and b) east-west transport components.

13, the steady state clearly presents itself in the area left behind by the radiating inertia-gravity waves already after this short time. Nevertheless, after sufficient time (here 150 days) the inertia-gravity waves have propagated out of the domain and only kinetic energy remains to achive the geostrophic balance. Thus, some of the initial potential energy is lost due to the inertia gravity waves and some is converted into kinetic energy, which explains why the bump is reduced in amplitude and wider in width. As shown by Figure 13 the energy contained in the inertia-gravity waves is small. Due to the application of

the FRS as an open boundary condition the energy carried by the inertia-gravity waves is not reflected at the western and eastern boundaries, but are damped within the FRS zones. This damping is a manifestation of the sponge-like behavior of the FRS. Since the northern and southern walls are closed the waves are reflected there and causes some noise in the results

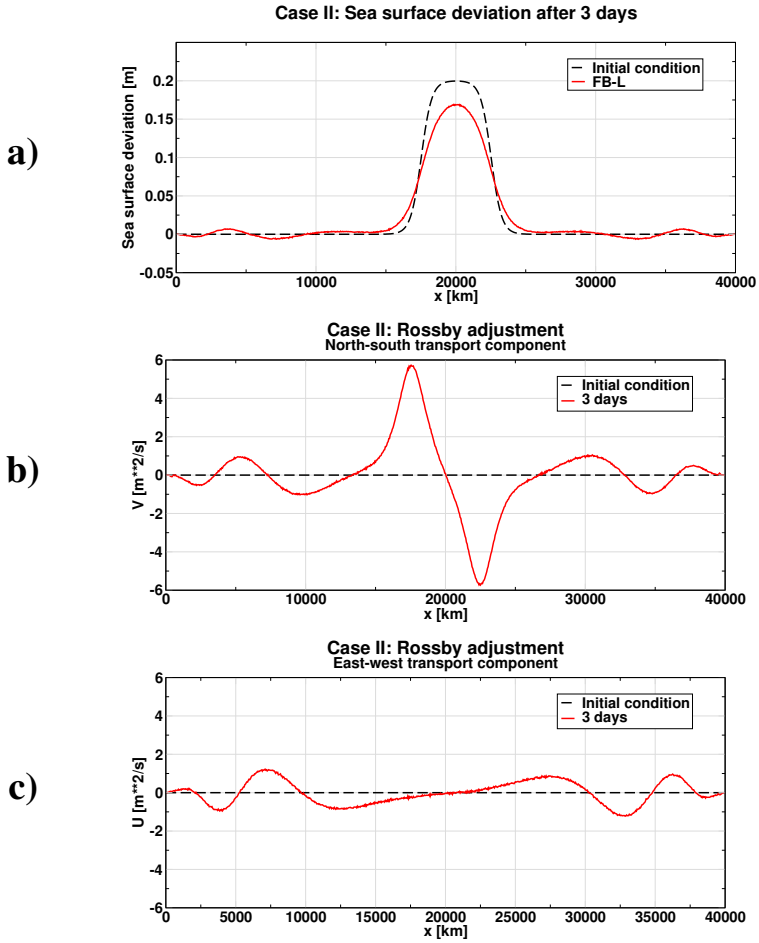

Figure 13: Case II: Solutions as a function of the east-west direction along $y = y_0 = 25000$ km after 3 days of integration. The results shown are derived using the linear FB-L scheme.

This case is identical to Case B of (cf. *Holm et al.*, 2020, their Figure 7), and hence the solutions may be compared directly with the solution they obtain using their four schemes. It is satisfying to observe that the solutions derived by them using the FBL and CTCS schemes are identical to the solutions shown by Figure 10 regarding the FB-L and CTCS-N schemes.

8.3 Cases III: Small Kelvin waves

As alluded to in Section 7.3 the Kelvin waves are moving with the boundary to its right on the northern hemisphere. In this case the boundary is the southern boundary, so the waves move eastward (to the right). If the Kelvin waves are linear they all propagate at a speed $c_0 = \sqrt{gH_0}$. Under these circumstances the initial state will not change in time as the Kelvin waves moves eastward. Thus, imposing a cyclic boundary conditions at the western and eastern boundary, a true linear solution will repeat itself when it has traveled a distance equal to the length L_x of the basin. The time it takes to travel this distance is $T = L_x/c_0$, which henceforth is referred to as one cycle or period. Plotting a true linear solution for the times $t_m = mT$ on top of each other, where $m = 1, 2, \ldots$ is the number of cycles into the integration, they will be indistinguishable from the initial solution. Ideally this should also be true regarding the two numerical solutions derived using the linear FB-L and CTCS-L schemes.

Solutions based on the integration of the three schemes after $m = 1$, 5 and 10 cycles, are displayed by Figure 14 together with the initial state. As expected the two linear schemes FB-L and CTCS-L provide for all practial purposes identical results. However, they do not completely overlay the initial state after, e.g., 10 cycles. As alluded to in Section 8.1 the schemes contain numerical dispersion. Thus signs of this dispersion in the form of trailing waves lagging behind the two slopes are evident. However, most of the waves are well resolved (small $\alpha\Delta x$), and hence they contain very little energy.

The results using the nonlinear CTCS-N are slightly different from the results applying the two linear schemes. A detailed inspection reveals that as time progresses the former has a tendency to steepen the slope slightly at the forefront and to similarly relax the hindslope. This gives the hump a somewhat tilted appearance after 10 periods in addition to showing signs of being numerically dispersive, in particular at the forefront.

The explanation of the tilting appearance is straightforward. While the linear Kelvin waves all propagate at the same speed, namely $c = c_0$, the nonlinear Kelvin waves propagate at speeds given by

$$c(x,y,t) = \sqrt{gh(x,y,t)} = \sqrt{g(H_0+\eta)}, \tag{88}$$

where $h(x,y,t)$ is the local depth of a water column and $\eta = \eta(x,y,t)$ is the local sea surface deviation (Figure 1). Thus, in contrast to the linear Kelvin waves the nonlinearity allows Kelvin waves to propagate at different speeds in accord with (88). Since $\eta \leq \eta_0$ the fastest waves propagate with a phase speed $c_{max} = \sqrt{g(H_0+\eta_0)}$ rather than c_0. Making use of the numbers displayed by Table 1 yields $c_{max} = 31.3288$ ms^{-1} and $c_0 = 31.3209$

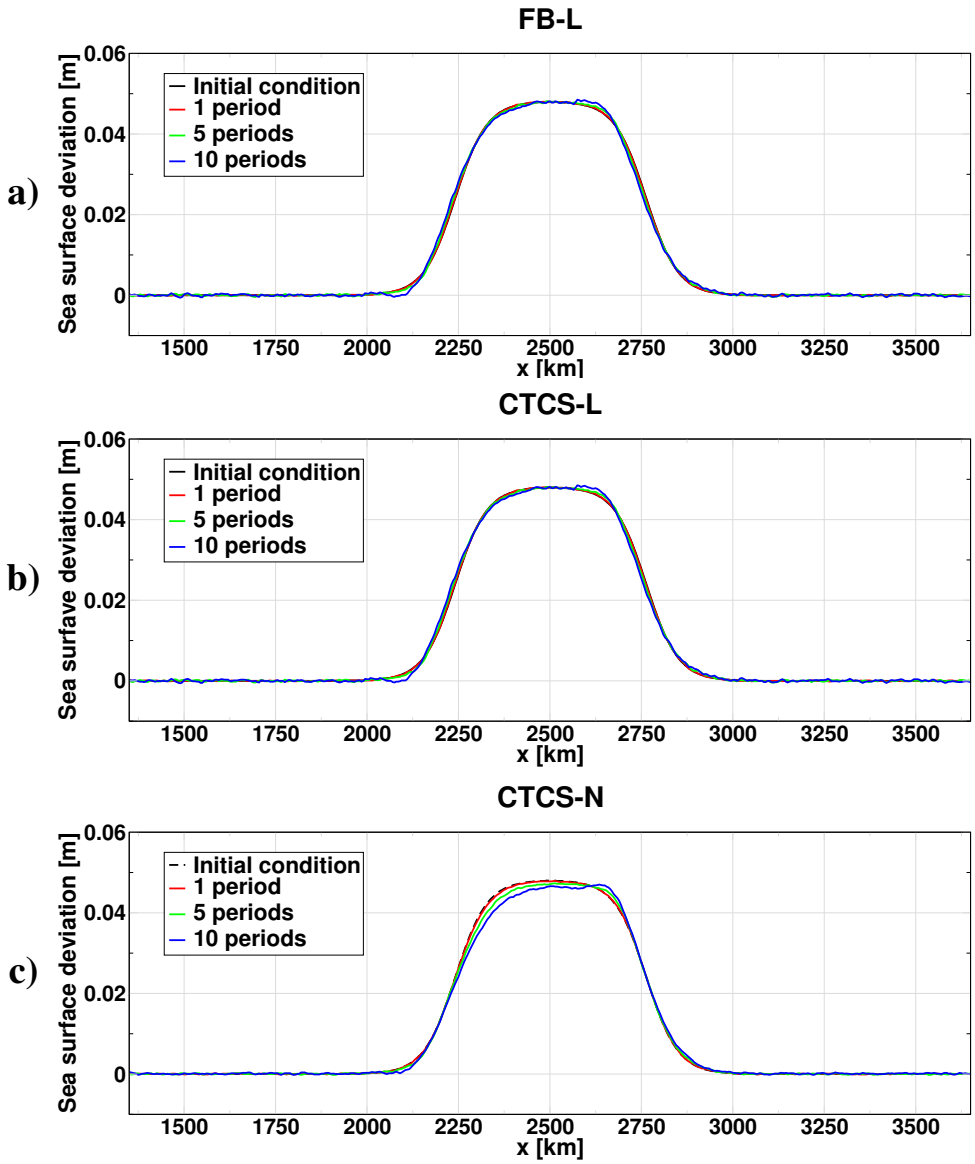

Figure 14: Case III: The sea surface deviation close to the southern boundary ($y = 0.5\Delta y$) employing the three schemes. Numbers along the horizontal axis indicate distance along the southern boundary. The full lenght of the domain is 5000 km, but is truncated at 1350 km and 3650 km, respectively.

ms^{-1}. So in this case the fastest waves are just propagating a tiny bit faster by a factor of 1.00025 than the ones traveling at speed c_0. Nevertheless, the implication is that after 10 periods the fastest Kelvin waves have propagated a distance $\approx$ 12 km longer than the ones propagating at speed c_0. Thus, even though the phase speed of the fastest Kelvin waves are just barely faster its impact after 10 periods, as revealed by Figure 14, is indeed

detectable in the CTCS-N solution. Thus, and as revealed in Section 8.4 regarding the large amplitude Kelvin waves, this tilting gets more pronounced when the amplitude of the hump is increased and thereby increases the speed of the fastest Kelvin waves.

Comparing the FB-L and CTCS-L solutions to those presented by *Holm et al.* (2020) (their Figure 11) it is satisfying to observe that they are nearly identical. Nevertheless, a closer inspection indicates that the solution derived employing the CTCS-N scheme differs slightly from the solution derived applying their CTCS scheme. One difference might be that no eddy viscosity was employed in our case (Table 1).

8.4 Cases IV: Large Kelvin waves

The respective solutions after $m = 1$, 5 and 10 periods for a large Kelvin wave with an initial amplitude of 2 meters are displayed by Figure 15. As expected the two linear schemes FB-L and CTCS-L provides solutions that are quite similar to one another. As before the speed of the Kelvin waves are restricted to be the same for all, namely $c = c_0 = \sqrt{gH_0}$. Hence thay all propagate with this speed regardless of what the sea surface deviation may be. So as in Case III the linear solutions provide a bump which is almost intact and symmetric even after 10 periods. There is, however, an assymetry which is easily detectable after 10 periods, and slightly more pronounced than in Case III. This assymetry is again attributed to the fact that the schemes contain numerical dispersion.

In contrast the CTCS-N scheme provides a solution which differs dramatically from the small Kelvin wave case and from the linear schemes. With reference to Section 8.3 and to Table 1 this is to be expected. The speed of the fastest Kelvin wave is now $c_{max} \approx 31.6326$ ms^{-1}, and hence it propagates faster by a factor of ~ 1.0097 than the slowest Kelvin wave propagating at speed c_0. Thus, already after 1 period the fastest Kelvin wave has propagated a distance approximately 50 km longer than the slowest one. The implication is that the faster Kelvin waves overtake the slower ones much swifter than in Case III. Thus, already after one cycle the forefront of the bump is pretty steep while the hindslope is noticeably relaxed, and the bump has a distinct tilted appearance. At this stage the steepness of the slope is still well enough resolved by the chosen grid size to avoid numerical dispersion to show. Inspection of Figure 16 though reveals that after two periods of integration this is no longer the case. At this time the fastest Kelvin waves have propagated a distance about 100 km longer than the slowest one. Hence, some time between one and two periods into the integration the steepness of the foreslope is no longer well resolved as a shock is formed. The numerical dispersion therefore shows

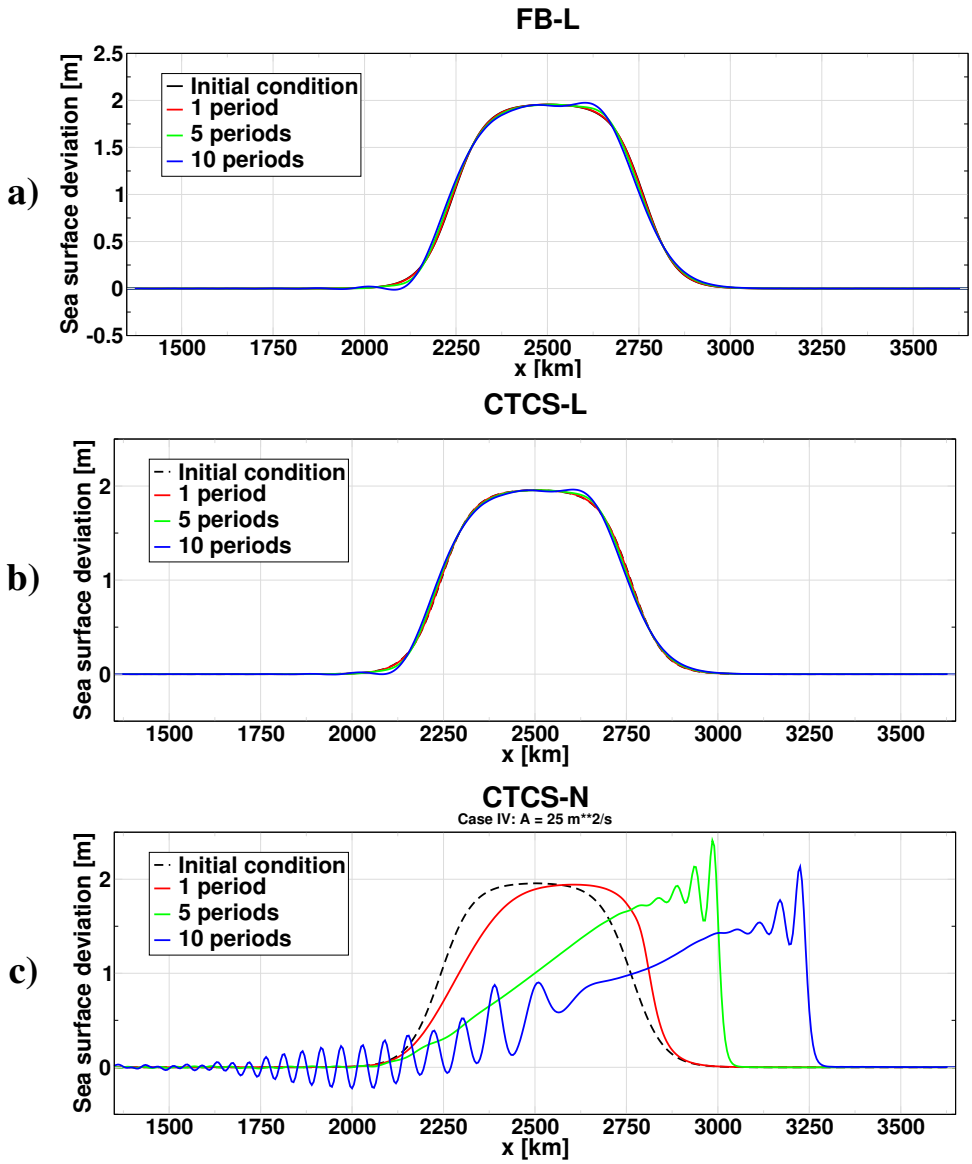

Figure 15: Case IV: As Figure 14, but for large Kelvin waves.

itself as a trailing wave lagging behind the shock, not unlike what happened in Case I. Moreover, once the shock is formed it moves with a speed $c_{max} > c_0$, and consequently the shock appears to lead the initial bump as displayed by Figures 15 and 16. This is again similar to the nonlinear Case I solution.

Furthermore, the nonlinear solution provided by the CTCS-N scheme disagree with the solution provided by *Holm et al.* (2020) (their Figure 12). It is also more in line with the solutions derived by using the KP and CDKLM schemes for the large Kelvin wave case. In fact, filtering the CTCS-N solution as shown by Figure 17 the result is

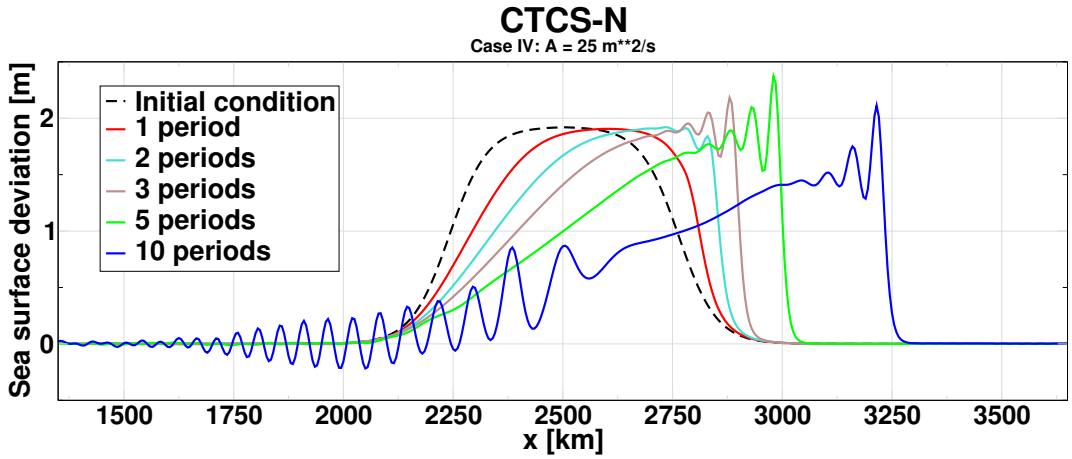

Figure 16: Case IV: The nonlinear solution after various periods. Otherwise as Figure 15.

indeed very similar to their solutions derived using the KP and CDKLM schemes (their Figure 12). Interestingly, the nonlinear CTCS solution of *Holm et al.* (2020) appears to be mirroring the CTCS-N solution, that is, the Kelvin waves are moving westward rather than eastward.

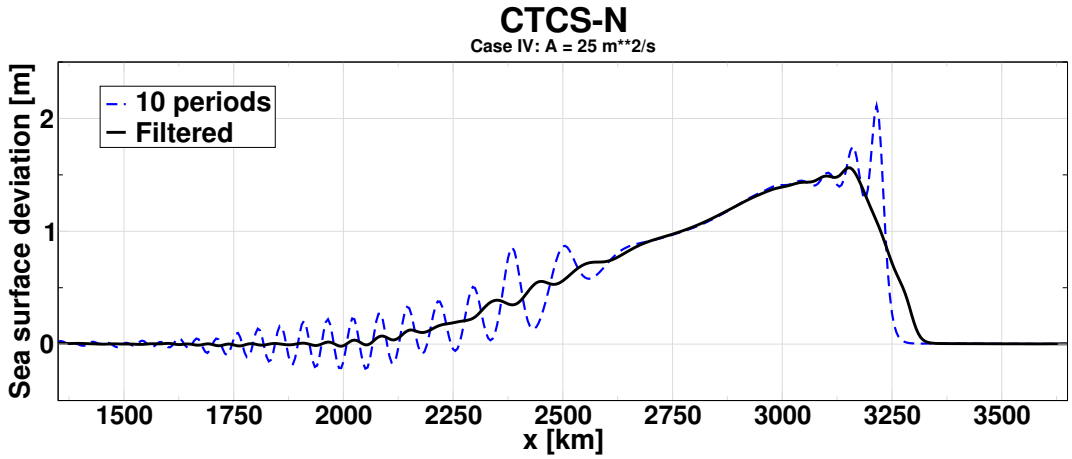

Figure 17: Case IV: The filtered sea surface deviation on top of the unfiltered one. The filter is an unweighted, moving average filter using 31 points. Otherwise as Figure 15.

It may be argued that increasing the resolution (decreasing the space increments) would improve the CTCS-N solution. However, since the CTCS-N scheme is a convergent scheme (*Røed*, 2019, Section 4.9, page 57) the numerical solution approaches the true solution only in the limit when the space increments and time step go to zero. Thus,

decreasing the space increments only postpones the onset of the waves due to numerical dispersion.

The numerical dispersion appearing as the shock forms (Figure 16) are indeed similar to the numerical dispersion experienced in Case I (Figure 8). However, in Case I the front was already established initially and the numerical dispersion in the form of trailing waves appeared immediately. In Case IV the shock establishes itself only after some time into the integration, and hence also the trailing waves caused by the numerical dispersion. Another similarity with Case I is that the amplitude of the dispersive waves grows in time and although the solution is stable even after 10 periods of integration it may eventually go unstable in a numerical sense later on.

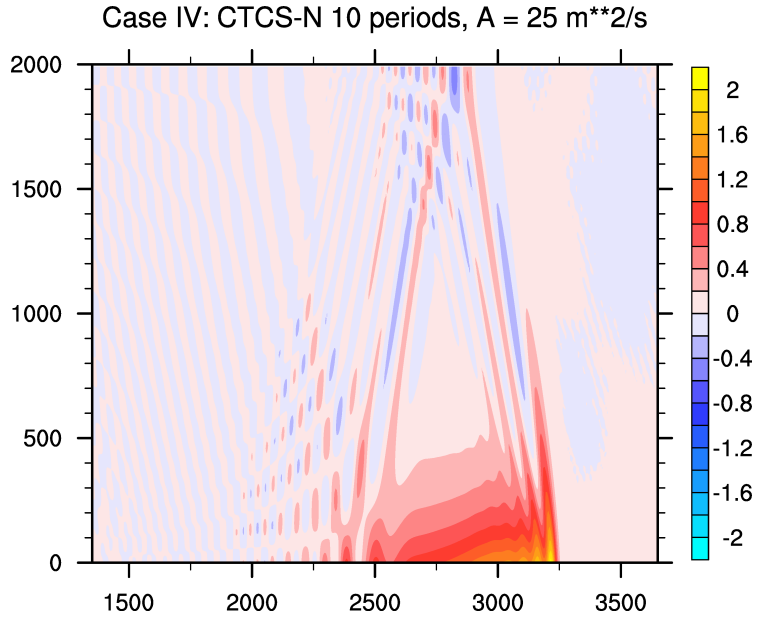

Figure 18: Case IV: The sea surface deviation after 10 periods of integration. A is the eddy viscosity coefficient. The distance in the east-west direction is truncated.

There are some differences though. Recall that in Case I the solution is one-dimensional, while in Case IV it is two-dimensional right from the start in that the amplitude of the initial bump decreases with increasing distance from the southern wall (Figure 7). As a result the fastest Kelvin wave some distance away from the southern wall moves slower than those closer to the southern wall. Thus, a longer and longer time will be needed in order for the fastest wave to overtake the slowest one as one proceed away from the southern wall. The dispersive waves therefore shows up first close to the southern wall and then progressivly later as one moves away from the wall. The two-dimensional impression is

therefore of waves that appears to be arching backwards away from the southern wall as depicted by Figure 18 after 10 periods of integration. At any specific time into the integration the amplitude of the waves are therefore decreasing away from the southern wall. In addition Case IV includes the effect of rotation that enables the bump to be balanced geostrophically.

Also present in the solution after 10 periods of integration is a striking wave like pattern lagging far behind the shock (Figures 16 and 18). As revealed by Figure 19 the depicted waves at the forefront appear to be reflected at the northern wall and to continue toward the southern wall. On their way to be reflected at the southern wall they disturb the tail of the solution which explains the trailing wave pattern.

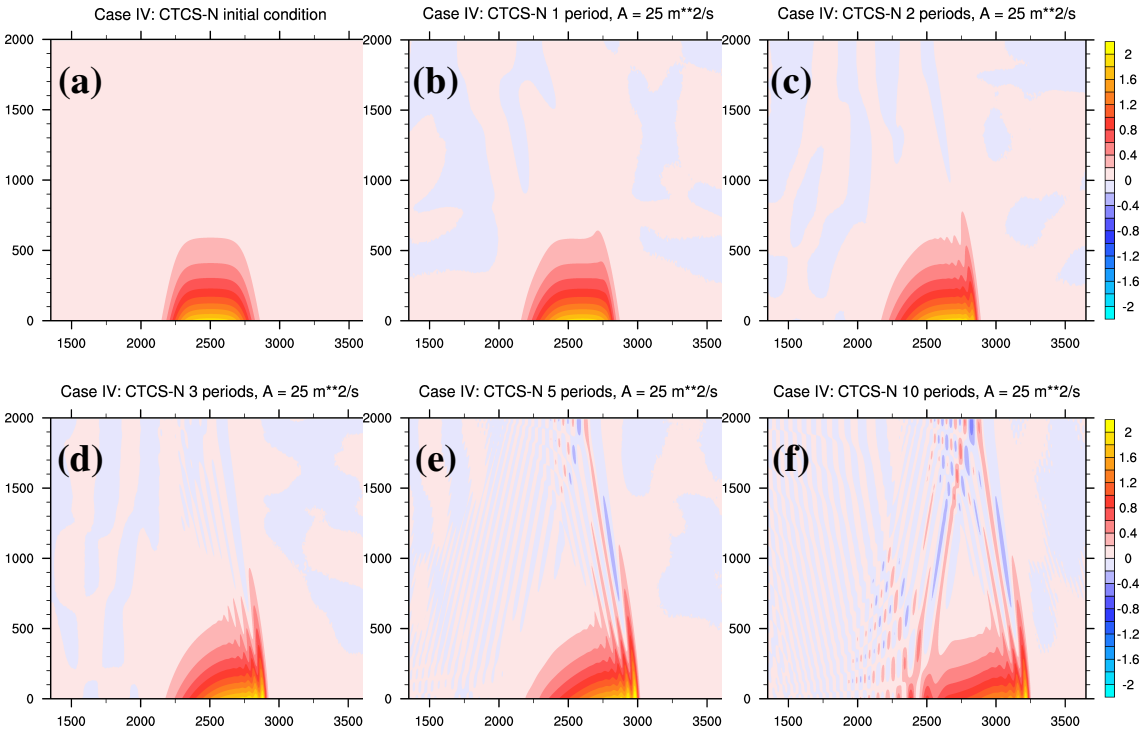

Figure 19: Case IV: As Figure 18, but showing the initial condition (a), the solution after 1 period (b), 2 periods (c), 3 periods (d), 5 periods (e), and 10 periods (f) of integration.

9 Summary and some final remarks

Considered are linear and nonlinear solutions to the shallow water equations for four cases, namely Case I: Dam break on a wet domain without rotation, Case II: Rossby

adjustment, Case III: Small Kelvin waves, and Case IV: Large Kelvin waves. These cases correspond to and are identical to four of the cases performed by *Holm et al.* (2020). Thus, a comparison of their results and those presented here are facilitated.

The cases are all run in a two-dimensional basin with a flat bottom oriented so that the x-axis runs from west to east and the y-axis from south to north (cf. Figures 2 and 3). The southern and northern boundaries are closed, reflecting walls, while the eastern and western boundaries are open. All cases are started from a specified initial state, which in Cases I and II is unbalanced. In Cases III and IV the initial state is balanced (geostrophic balance).

Two of the schemes studied here, namely the linear FB-L scheme and the nonlinear CTCS-N scheme are identical to two of the schemes investigated by *Holm et al.* (2020), specifically the schemes they denoted FBL and CTCS. In addition solutions derived using a linear version of the CTCS-N scheme, called the CTCS-L scheme, is provided for comparison with the linear FB-L scheme. All the above mentioned schemes are finite difference schemes. However, *Holm et al.* (2020) also included two finite volume schemes, namely the KP scheme and the CDKLM scheme credited to *Kurganov and Petrova* (2007) and *Chertok et al.* (2018), respectively.

In all four cases the linear FB-L and CTCS-L solutions are for all practical purposes identical to each other and to the linear (FBL) solution of *Holm et al.* (2020). In Cases II and III this is also true concerning the nonlinear CTCS-N scheme and the similar nonlinear CTCS scheme of *Holm et al.* (2020). However, in Cases I and IV they differ radically. Both of these cases contain a shock. Since the finite difference schemes used here contain numerical dispersion expected trailing waves appear in both of these cases in the solutions provided by the CTCS-N scheme. However, these dispersive waves are not present in the CTCS solutions of *Holm et al.* (2020) in Case I. Moreover, except for the expected dispersive waves, the Case I solution provided by the CTCS-N scheme is more in line with the solution derived using the KP and CDKLM schemes of *Holm et al.* (2020) (their figure 6) and the analytic solution. Regarding Case IV, which treats the highly nonlinear Kelvin wave case, the CTCS-N solution after one cycle is almost identical to the CDKLM solution of *Holm et al.* (2020). At this stage in the integration the slope of the front, which later forms a shock, is still well resolved and the numerical dispersion negligible. However, as the shock forms between cycle one and two the numerical dispersion becomes evident in the form of trailing waves of growing amplitudes lagging behind the shock. On the northern hemisphere the Kelvin waves moves with the boundary to its right, that is

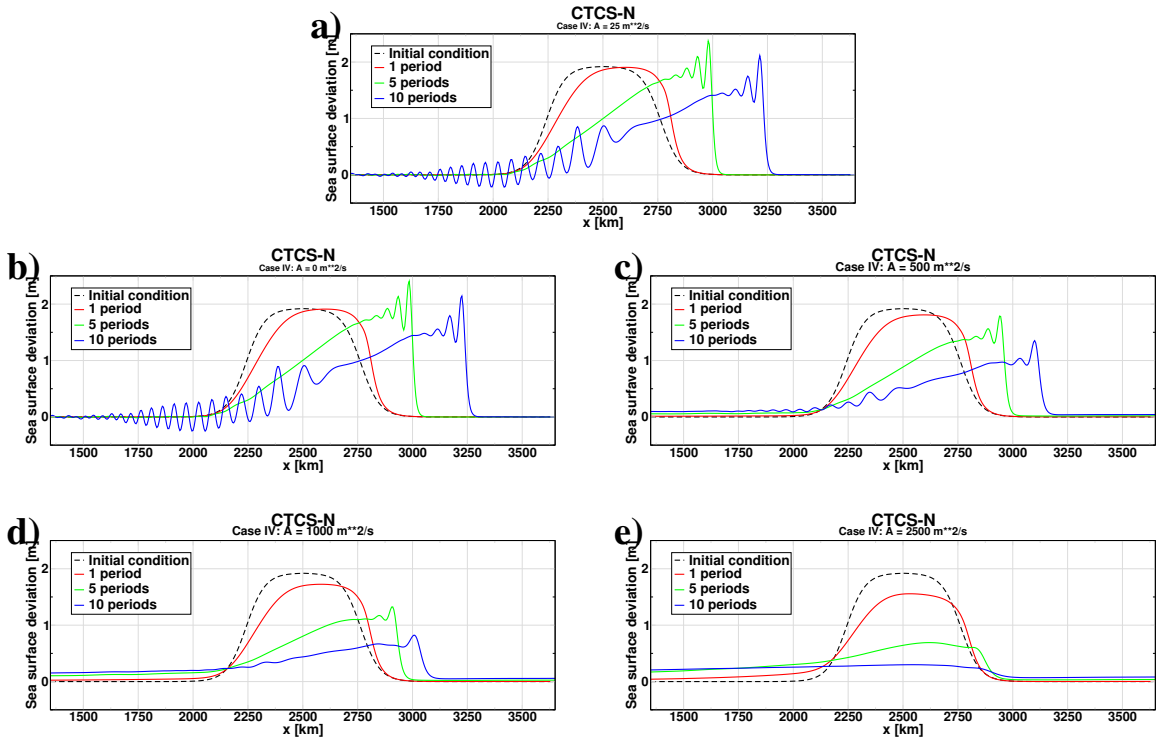

Figure 20: Case IV: The sea surface deviation for various choices of the eddy viscosity coefficient A (unit m^2/s). a) $A = 25$, b) $A = 0$, c) $A = 500$, d) $A = 1000$, and e) $A = 2500$.

eastward in Case IV. This is the case regarding both the CTCS-N solution studied here and the solutions derived using the finite volume schemes KP and CDKLM studied by *Holm et al.* (2020). Also looking at the CTCS solution of *Holm et al.* (2020) the Kelvin waves do appear to be moving eastward, but the tilting of the dome is opposite to that of all the other nonlinear schemes. Interestingly the CTCS solution of *Holm et al.* (2020) do contain the trailing dispersive waves in Case IV where the shock forms.

It may be argued that by increasing the eddy viscosity the dispersive waves my be damped and thereby produce a result more akin to the filtered solution shown by Figure 17, and hence more in line with the KP and CDKLM solutions of *Holm et al.* (2020). An attempt in this direction is displayed by Figure 20. As portrayed an increase in the eddy viscosity does damp the amplitude of the trailing dispersive waves. However, at the same time the speed of the front is slowed down and the amplitude of the dominant Kelvin

waves are decreased. Thus, increasing the eddy viscosity does not make the solution look more like the results using finite volume schemes KP and CDKLM. Thus, these more recent and modern finite volume schemes are superior to the finite difference schemes in the sense that they avoid the numerical dispersion, and thereby improves the ability to handle steep fronts and shocks numerically.

Nevertheless, the finite difference, nonlinear solution presented by *Holm et al.* (2020) (their CTCS scheme) is dramatically different from the CTCS-N solutions for Case I and IV presented in the present study, and also dramatically different from their solutions using the finite volume schemes KP and CDKLM. *Holm et al.* (2020) ascribes this difference being due to numerical dispersion. This is contrary to the conclusion reached and discussed in Section 8.4. There it was concluded that the scheme CTCS-N provided solutions more similar to the KP and CDKLM solutions in both Case I and Case IV. It is therefore concluded that the erratic behaviour of the CTCS scheme being due to numerical dispersion is misguided. Rather something must be incorrect in their GPU implementation of the CTCS scheme. Nonetheless the main conclusion reached by *Holm et al.* (2020) stands to be correct, namely that the KP and CDKLM are superior to the finite difference schemes regarding resolving steep fronts and shocks.

Acknowledgements

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A The FORTRAN program

The FORTRAN program consists of the main program `nonlinear_swe` and eight subroutines `init`, `savef`, `store`, `ctcs-n`, `euler-n`, `ctcs-l`, `euler-l` and `fbl`. The subroutines `ctcs-n`, `ctcs-l` and `fbl` contains the algorithms for the the three schemes CTCS-N, CTCS-L and FB-L, as their names suggest. The two additional schemes are linear and nonlinear versions of the Euler schemes nessecary to avoid the initial problem of the CTCS-N and CTCS-L schemes (cf. Sections 5.2 and 5.3). All subroutines are called from the main program `nonlinear_swe`.

A.1 The main program `nonlinear_swe.f90`

```
program nonlinear_swe
  implicit none
!-----------------------------------------------------------------
! Front matter
  ! Declarations:
     ! Assign logical unit numbers (lun) of files
  integer, parameter :: olunh1 = 11 ! outfileh1
  integer, parameter :: olunh2 = 12 ! outfileh2
  integer, parameter :: olunh3 = 13 ! outfileh3
  integer, parameter :: olunh4 = 14 ! outfileh4
  integer, parameter :: olunhe = 15 ! outfilehe
  integer, parameter :: olunhp = 16 ! outfilehp
  integer, parameter :: olunu1 = 21 ! outfileu1
  integer, parameter :: olunu2 = 22 ! outfileu2
  integer, parameter :: olunu3 = 23 ! outfileu3
  integer, parameter :: olunu4 = 24 ! outfileu4
  integer, parameter :: olunue = 25 ! outfileue
  integer, parameter :: olunup = 26 ! outfileup
  integer, parameter :: olunv1 = 31 ! outfilev1
  integer, parameter :: olunv2 = 32 ! outfilev2
  integer, parameter :: olunv3 = 33 ! outfilev3
  integer, parameter :: olunv4 = 34 ! outfilev4
  integer, parameter :: olunve = 35 ! outfileve
  integer, parameter :: olunvp = 36 ! outfilevp
  integer, parameter :: olunhov = 40 ! outfilehov
```

```
integer, parameter :: llun   = 41 ! logfile
   ! Space and time dimensions
integer, parameter :: np   = 4 ! # times variables saved
integer            :: jmax     ! # of cells along x-axis
integer            :: kmax     ! # of cells along y-axis
   ! Runtime parameters (independent of case)
integer, parameter :: Ng  = 10     ! # space increments FRS
integer, parameter :: nfl = 5000   ! time steps in one cycle
integer, parameter :: nhov = 50    ! # time steps Hovmøller
real, parameter    :: g = 9.81    ! Gravitational acceleration
real, parameter    :: ff = 1.2E-04 ! Coriolis parameter
   ! Runtime constants (dependent on case)
integer :: n2    ! # of time steps to complete one cycles
integer :: n3    ! # of time steps to complete five cycles
integer :: n4    ! # of time steps to complete ten cycles
integer :: ntmax ! Maximum # of time steps
real    :: A     ! Eddy viscosity coefficient [m**2/s]
real    :: f     ! Coriolis parameter [1/s]
real    :: dx    ! Space increment along x-axis [m]
real    :: dy    ! Space increment along y-axis [m]
real    :: x0    ! Center of maximum initial eta [m]
real    :: y0    ! Location of maximum eta
real    :: H0    ! Equilibrium depth (flat bottom) [m]
real    :: c0    ! Phase speed of linear Kelvin wave [m/s]
real    :: Lx    ! Length of basin east-west direction
real    :: Ly    ! Length of basin north-south direction
real    :: dt    ! Time step [s]
real    :: dt2   ! Twice the time step
real    :: eta0  ! Initial maximum sea surface deviation [m]
real    :: Lr    ! Rossby's deformation radius [m]
integer :: kp    ! Cell # at which Up,Vp,hp, & etap are saved
   ! Auxiliary variables
integer :: nt                   ! # of time levels of scheme
real    :: time                 ! Time in seconds
```

```
real    :: htime                ! Time in hours
real    :: dtime                ! Time in days
integer :: n,ntime              ! Time step counters
integer :: iold,icur,inew,isave ! Time level counters
integer :: j,k                  ! Space increment (j,k)
integer :: jp1                  ! Auxiliary counter
integer :: nsave                ! # of time steps to save
   ! Auxiliary runtime constants used in subroutine update
real :: pru,prv       ! Constants used in pressure terms
real :: ax,ay,a4x,a4y ! Constants used in nonlinear terms
real :: mixu,mixv     ! Constants used in mixing terms
real :: R             ! Constant due to Dufort-Frankel
real :: corU,fdt      ! Help factors Coriolis term
   ! Auxiliary variables used in subroutine update:
real :: CU,CV                          ! Coriolis terms
real :: PU,PV                          ! Pressure terms
real :: AUx,AUy,AVx,AVy                ! Advection terms
real :: MU,MV                          ! Eddy viscosity terms
real :: Uhjpk,Uhjk,UAjk,VAjk,hAjk      ! Help variables
real :: UAjkm,VAjkm,hAjkm              ! Help variables
real :: UAjmk,VAjmk,hAjmk,Vhjkp,Vhjk ! Help variables
   ! Auxiliary constants and arrays used in subroutine init:
real :: arg11,arg12 ! Help variables
real :: argx,argy   ! Help variables
real :: arg4,LL,DD  ! Help constants
   ! Dimension of allocateable arrays:
      ! Thickness points x-axis in meters [m]
real, allocatable, dimension (:)                    :: x
      ! Thickness points x-axis in kilometers [km]
real, allocatable, dimension (:)                    :: xkm
      ! Thickness points along y-axis [m]
real, allocatable, dimension (:)                    :: y
      ! Thickness points along y-axis [km]
real, allocatable, dimension (:)                    :: ykm
```

```
      ! Initial left-hand depth Case A [m]
real, allocatable, dimension (:)                  :: Hl
      ! Initial right-hand depth Case A [m]
real, allocatable, dimension (:)                  :: Hr
      ! Offshore transport [m**2/s]
real, allocatable, dimension (:,:,:)              :: U
      ! As U, but at specified times [m**2/s]
real, allocatable, dimension (:,:,:)              :: Ue
      ! As Ue, but at auxiliary points [m**2/s]
real, allocatable, dimension (:,:,:)              :: Up
      ! Alongshore transport [m**2/s]
real, allocatable, dimension (:,:,:)              :: V
      ! As V, but at specified times [m**2/s]
real, allocatable, dimension (:,:,:)              :: Ve
      ! As Ve, but at auxiliary points [m**2/s]
real, allocatable, dimension (:,:,:)              :: Vp
      ! Thickness of water column [m]
real, allocatable, dimension (:,:,:)              :: h
      ! As h, but at specified times [m**2/s]
real, allocatable, dimension (:,:,:)              :: he
      ! As he, but at auxiliary points [m]
real, allocatable, dimension (:,:,:)              :: hp
      ! Sea surface deviation from equilibrium [m]
real, allocatable, dimension (:,:,:)              :: eta
      ! As eta, but at the specified times [m]
real, allocatable, dimension (:,:,:)              :: etae
      ! As etae, but at auxiliary points [m]
real, allocatable, dimension (:,:,:)              :: etap
   ! Allocateable auxiliary arrays used in subroutine init:
real, allocatable, dimension(:) :: alf  ! Relax para in x
real, allocatable, dimension(:) :: alfl ! Relax para left in x
real, allocatable, dimension(:) :: alfr ! Relax para right
real, allocatable, dimension(:) :: arg1 ! y dependence
real, allocatable, dimension(:) :: arg2 ! x dependence
```

```
real, allocatable, dimension(:) :: arg3 ! As arg1
real, allocatable, dimension(:) :: arg5 ! As arg1
real, allocatable, dimension(:) :: corV ! Factor Coriolis term

   ! Flags
integer :: icase   ! 1,2,3,4 => Cases I,II,III,IV
integer :: ischeme ! 1,2,3 => Schemes FB-L, CTCS-L,CTCS-N
integer :: ishap   ! 0,1 => no filter, Shapiro filter applied
integer :: iev     ! 0,1 => No eddy viscosity (EV), EV applied
integer :: ibc     ! 1,2 => Cyclic,FRS boundary condition

   ! Character strings:
character(LEN=80) :: outfileh1  ! 1st (initial) etae field
character(LEN=80) :: outfileu1  ! 1st (initial) Ue field
character(LEN=80) :: outfilev1  ! 1st (initial) Ve field
character(LEN=80) :: outfileh2  ! 2nd etae field
character(LEN=80) :: outfileu2  ! 2nd Ue field
character(LEN=80) :: outfilev2  ! 2nd Ve field
character(LEN=80) :: outfileh3  ! 3rd etae field
character(LEN=80) :: outfileu3  ! 3rd Ue field
character(LEN=80) :: outfilev3  ! 3rd Ve field
character(LEN=80) :: outfileh4  ! 4th etae field
character(LEN=80) :: outfileu4  ! 4th Ue field
character(LEN=80) :: outfilev4  ! 4th Ve field
character(LEN=80) :: outfilehe  ! etae along x (k=2)
character(LEN=80) :: outfileue  ! Ue along x (k=2)
character(LEN=80) :: outfileve  ! Ve along x (k=2)
character(LEN=80) :: outfilehp  ! etap along x (k=2)
character(LEN=80) :: outfileup  ! Up along x (k=2)
character(LEN=80) :: outfilevp  ! Vp along x (k=2)
character(LEN=80) :: outfilehov ! Hovmoller file
character(LEN=80) :: logfile    ! Holds the logrun info
integer           :: res        ! Status file operations
```

```
  ! End declarations
!-----------------------------------------------------------------
  ! Assign meaningful names to the outfiles and the logfile
  outfileh1  = "Initial_eta-results.dat"
  outfileu1  = "Initial_u-results.dat"
  outfilev1  = "Initial_v-results.dat"
  outfileh2  = "2nd_eta-results.dat"
  outfileu2  = "2nd_u-results.dat"
  outfilev2  = "2nd_v-results.dat"
  outfileh3  = "3rd_eta-results.dat"
  outfileu3  = "3rd_u-results.dat"
  outfilev3  = "3rd_v-results.dat"
  outfileh4  = "4th_eta-results.dat"
  outfileu4  = "4th_u-results.dat"
  outfilev4  = "4th_v-results.dat"
  outfilehe  = "etae-results.dat"
  outfileue  = "ue-results.dat"
  outfileve  = "ve-results.dat"
  outfilehp  = "etap-results.dat"
  outfileup  = "up-results.dat"
  outfilevp  = "vp-results.dat"
  outfilehov = "Hovmoller-results.dat"
  logfile    = "log.dat"
!-----------------------------------------------------------------
  ! Open files
     ! 1: Files related to eta
  open(unit=olunh1,file=outfileh1,form="formatted",iostat=res)
  if(res /= 0) then                  ! An error has occured
     print*, "Error in opening outfileh1, status: ", res
     stop                            ! Stop the program
  endif
  open(unit=olunh2,file=outfileh2,form="formatted",iostat=res)
  if(res /= 0) then                  ! An error has occured
      print*, "Error in opening outfileh2, status: ", res
```

```
    stop                          ! Stop the program
endif
open(unit=olunh3,file=outfileh3,form="formatted",iostat=res)
if(res /= 0) then                 ! An error has occured
   print*, "Error in opening outfileh3, status: ", res
   stop                           ! Stop the program
endif
open(unit=olunh4,file=outfileh4,form="formatted",iostat=res)
if(res /= 0) then                 ! An error has occured
   print*, "Error in opening outfileh4, status: ", res
   stop                            ! Stop the program
endif
open(unit=olunhe,file=outfilehe,form="formatted",iostat=res)
if(res /= 0) then                 ! An error has occured
    print*, "Error in opening outfilehe, status: ", res
    stop                          ! Stop the program
endif
open(unit=olunhp,file=outfilehp,form="formatted",iostat=res)
if(res /= 0) then                 ! An error has occured
   print*, "Error in opening outfilehp, status: ", res
   stop                           ! Stop the program
endif
   ! 2: Files related to U
open(unit=olunu1,file=outfileu1,form="formatted",iostat=res)
if(res /= 0) then                 ! An error has occured
   print*, "Error in opening outfileu1, status: ", res
   stop                           ! Stop the program
endif
open(unit=olunu2,file=outfileu2,form="formatted",iostat=res)
if(res /= 0) then                 ! An error has occured
   print*, "Error in opening outfileu2, status: ", res
   stop                           ! Stop the program
endif
open(unit=olunu3,file=outfileu3,form="formatted",iostat=res)
```

```
if(res /= 0) then                  ! An error has occured
   print*, "Error in opening outfileu3, status: ", res
   stop                            ! Stop the program
endif
open(unit=olunu4,file=outfileu4,form="formatted",iostat=res)
if(res /= 0) then                  ! An error has occured
   print*, "Error in opening outfileu4, status: ", res
   stop                            ! Stop the program
endif
open(unit=olunue,file=outfileue,form="formatted",iostat=res)
if(res /= 0) then                  ! An error has occured
   print*, "Error in opening outfileue, status: ", res
   stop                            ! Stop the program
endif
open(unit=olunup,file=outfileup,form="formatted",iostat=res)
if(res /= 0) then                  ! An error has occured
   print*, "Error in opening outfileup, status: ", res
   stop                            ! Stop the program
endif
   ! 3: Files related to V
open(unit=olunv1,file=outfilev1,form="formatted",iostat=res)
if(res /= 0) then                  ! An error has occured
   print*, "Error in opening outfilev1, status: ", res
   stop                            ! Stop the program
endif
open(unit=olunv2,file=outfilev2,form="formatted",iostat=res)
if(res /= 0) then                  ! An error has occured
   print*, "Error in opening outfilev2, status: ", res
   stop                            ! Stop the program
endif
open(unit=olunv3,file=outfilev3,form="formatted",iostat=res)
if(res /= 0) then                  ! An error has occured
   print*, "Error in opening outfilev3, status: ", res
   stop                            ! Stop the program
```

```
  endif
  open(unit=olunv4,file=outfilev4,form="formatted",iostat=res)
  if(res /= 0) then                  ! An error has occured
     print*, "Error in opening outfilev4, status: ", res
     stop                            ! Stop the program
  endif
  open(unit=olunve,file=outfileve,form="formatted",iostat=res)
  if(res /= 0) then                  ! An error has occured
     print*, "Error in opening outfileve, status: ", res
     stop                            ! Stop the program
  endif
  open(unit=olunvp,file=outfilevp,form="formatted",iostat=res)
  if(res /= 0) then                  ! An error has occured
     print*, "Error in opening outfilevp, status: ", res
     stop                            ! Stop the program
  endif
  open(unit=olunhov,file=outfilehov,form="formatted",iostat=res)
  if(res /= 0) then                  ! An error has occured
     print*, "Error in opening outfilehov, status: ", res
     stop                            ! Stop the program
  endif
  ! End open files
!-----------------------------------------------------------------
  ! Set flags
  write(6,*) 'Choose case to run (icase):'
  write(6,*) '1 => Case A: Breaking of a dam, no rotation'
  write(6,*) '2 => Case B: Adjustment under gravity, rotating'
  write(6,*) '3 => Case C: Small Kelvin waves'
  write(6,*) '4 => Case D: Large Kelvin waves'
  read(5,*) icase
  if( (icase > 4).or.(icase < 1) ) then
      stop 'icase out of bounds. Start over.'
  endif
  if( (icase == 1).or.(icase == 2) ) then  ! Case I and II
```

```
      ibc = 2                ! FRS at east-west boundaries
    else                                    ! Case III and IV
      ibc = 1                ! Cyclic east-west
    endif
    write(6,*) 'Choose scheme (ischeme):'
    write(6,*) '1 => FB-L, 2 => CTCS-L, 3 => CTCS-N)'
    read(5,*) ischeme
    if( (ischeme > 3).or.(ischeme < 1) ) then
      stop 'ischeme out of bounds. Start over.'
    endif
    if(ischeme == 1) then          ! FB-L
      iev = 0                  ! No eddy viscosity
    else                           ! CTCS-L and CTCS-N
      write(6,*) 'Choose whether to apply Eddy Viscosity (iev):'
      write(6,*) '0 => no EV, 1 => apply EV'
      read(5,*) iev
      if( (iev > 1).or.(iev < 0) ) then
        stop 'iev out of bounds. Start over.'
      endif
    endif
    write(6,*) 'Apply Shapiro filter (ishap)? 0 => No, 1 => Yes'
    read(5,*) ishap
    if( (ishap > 1).or.(ishap < 0) ) then
      stop 'ishap out of bounds. Start over.'
    endif
    ! End set flags
!-----------------------------------------------------------------
    ! Specify # of time levels in scheme
    if(ischeme == 1) then
      nt = 2           ! FB-L has only two time levels
    else
      nt = 3           ! The 2 CTCS schemes have three time levels
    endif
!-----------------------------------------------------------------
```

```
  ! Specify # of cells (including ghost cells) in x and y
  ! directions and time levels for saving of variables
  if(icase == 1) then     ! Case I: Breaking of a dam (wet)
     jmax = 501
     kmax = 52
     kp   = 2
     n2   = 200 ! Final = 2 s
     n3   = 400 ! Final = 4 s
     n4   = 600 ! Final = 6 s
     ntmax= n4 + 1
  elseif(icase == 2) then ! Case II: Adjustment under gravity
     jmax = 801
     kmax = 1002
     kp   = 501
     n2   = 864    ! = 1 day
     n3   = 1728   ! = 3 days
    !n3   = 64800  ! = 75 days
    !n4   = 129600 ! = 150 days
ntmax= n4 + 1
  else                    ! Case III/IV: Small/Large Kelvin waves
    jmax = 1001
    kmax = 202
    kp   = 2
   !n2   = nfl    ! Testing
   !n3   = 2*nfl  ! Testing
   !n4   = 3*nfl  ! Testing
    n2   = nfl   ! Final = # to complete one cycles
    n3   = 5*nfl  ! Final = # to complete five cycles
    n4   = 10*nfl ! Final = # to complete ten cycles
ntmax= n4 + 1
  endif
!-------------------------------------------------------------
  ! Allocate space for variables
  allocate (x(jmax), STAT=res)
```

```
if(res /= 0) then
   print*, 'Error in allocating space for x, status: ', res
   stop
endif
allocate (xkm(jmax), STAT=res)
if(res /= 0) then
   print*, 'Error in allocating space for xkm, status: ', res
   stop
endif
allocate (y(kmax), STAT=res)
if(res /= 0) then
   print*, 'Error in allocating space for y, status: ', res
   stop
endif
allocate (ykm(kmax), STAT=res)
if(res /= 0) then
   print*, 'Error in allocating space for ykm, status: ', res
   stop
endif
allocate (Hl(kmax), STAT=res)
if(res /= 0) then
   print*, 'Error in allocating space for y, status: ', res
   stop
endif
allocate (Hr(kmax), STAT=res)
if(res /= 0) then
   print*, 'Error in allocating space for y, status: ', res
   stop
endif
allocate (U(jmax,kmax,nt), STAT=res)
if(res /= 0) then
   print*, 'Error in allocating space for U, status: ', res
   stop
endif
```

```
allocate (Ue(jmax,kmax,np), STAT=res)
if(res /= 0) then
   print*, 'Error in allocating space for Ue, status: ', res
   stop
endif
allocate (Up(jmax,kmax,np), STAT=res)
if(res /= 0) then
   print*, 'Error in allocating space for Up, status: ', res
   stop
endif
allocate (V(jmax,kmax,nt), STAT=res)
if(res /= 0) then
   print*, 'Error in allocating space for V, status: ', res
   stop
endif
allocate (Ve(jmax,kmax,np), STAT=res)
if(res /= 0) then
   print*, 'Error in allocating space for Ve, status: ', res
   stop
endif
allocate (Vp(jmax,kmax,np), STAT=res)
if(res /= 0) then
   print*, 'Error in allocating space for Vp, status: ', res
   stop
endif
allocate (h(jmax,kmax,nt), STAT=res)
if(res /= 0) then
   print*, 'Error in allocating space for h, status: ', res
   stop
endif
allocate (he(jmax,kmax,np), STAT=res)
if(res /= 0) then
   print*, 'Error in allocating space for he, status: ', res
   stop
```

```
endif
allocate (hp(jmax,kmax,np), STAT=res)
if(res /= 0) then
   print*, 'Error in allocating space for hp, status: ', res
   stop
endif
allocate (eta(jmax,kmax,nt), STAT=res)
if(res /= 0) then
   print*, 'Error in allocating space for eta, status: ', res
   stop
endif
allocate (etae(jmax,kmax,np), STAT=res)
if(res /= 0) then
   print*, 'Error in allocating space for etae, status: ', res
   stop
endif
allocate (etap(jmax,kmax,np), STAT=res)
if(res /= 0) then
   print*, 'Error in allocating space for etap, status: ', res
   stop
endif
allocate (corV(kmax), STAT=res)
if(res /= 0) then
   print*, 'Error in allocating space for corV, status: ', res
   stop
endif
allocate (alf(jmax), STAT=res)
if(res /= 0) then
   print*, 'Error in allocating space for alf, status: ', res
   stop
endif
allocate (alfl(jmax), STAT=res)
if(res /= 0) then
   print*, 'Error in allocating space for alfl, status: ', res
```

```
      stop
    endif
    allocate (alfr(jmax), STAT=res)
    if(res /= 0) then
      print*, 'Error in allocating space for alfr, status: ', res
      stop
    endif
    ! End allocating variables
!-----------------------------------------------------------------
    ! Specify runtime parameters and constants
    if(icase == 1) then                ! Case I
      f    = 0.*ff       ! [1/s]
      dx   = 0.02        ! [m]
      dy   = 0.02        ! [m]
      Lx   = (jmax-1)*dx ! [m]
      Ly   = (kmax-2)*dy ! [m]
      x0   = Lx/2.       ! [m]
      y0   = Ly/2.       ! [m]
      H0   = 0.003       ! [m]
      A    = 0.1 ! [m**2/s]
      c0   = sqrt(g*H0) ! [m/s]
      dt   = 0.01        ! [s]
      eta0 = 0.002       ! [m]
      Lr   = 0.0 ! [m]
    elseif(icase == 2) then             ! Case II
      f    = ff ! [1/s]
      dx   = 5.0E+4      ! [m] = 50 km
      dy   = 5.0E+4      ! [m] = 50 km
      Lx   = (jmax-1)*dx ! [m]
      Ly   = (kmax-2)*dy ! [m]
      x0   = Lx/2.       ! [m]
      y0   = Ly/2.       ! [m]
      H0   = 1.0E+3      ! [m] = 1 km
      A    = 25. ! [m**2/s]
```

```
   c0   = sqrt(g*H0) ! [m/s]
   dt   = 100.       ! [s]
   Lr   = c0/f       ! [m]
   eta0 = 0.2 ! [m]
   Hl   = H0 ! [m]
   Hr   = H0 ! [m]
elseif(icase == 3) then           ! Case III
   f    = ff ! [1/s]
   dx   = 5.0E+3     ! [m]
   dy   = 1.0E+4     ! [m]
   Lx   = (jmax-1)*dx ! [m]
   Ly   = (kmax-2)*dy ! [m]
   x0   = Lx/2.      ! [m]
   y0   = -0.5*dy    ! [m]
   H0   = 1.0E+2     ! [m]
   A    = 25. ! [m**2/s]
   c0   = sqrt(g*H0) ! [m/s]
   dt   = Lx/(nfl*c0) ! [s]
   Lr   = c0/f       ! [m]
   eta0 = 0.05       ! [m]
   Hl   = H0 ! [m]
   Hr   = H0 ! [m]
else                              ! Case IV
   f    = ff ! [1/s]
   dx   = 5.0E+3     ! [m]
   dy   = 1.0E+4     ! [m]
  !dx   = 2.5E+3     ! [m], Half mesh size
  !dy   = 0.5E+4     ! [m], Half mesh size
   Lx   = (jmax-1)*dx ! [m]
   Ly   = (kmax-2)*dy ! [m]
   x0   = Lx/2.      ! [m]
   y0   = -0.5*dy    ! [m]
   H0   = 1.0E+2     ! [m]
   A    = 25. ! [m**2/s]
```

```
      c0   = sqrt(g*H0) ! [m/s]
      dt   = Lx/(nfl*c0) ! [s]
      Lr   = c0/f        ! [m]
      eta0 = 2.0 ! [m]
      Hl   = H0 ! [m]
      Hr   = H0 ! [m]
    endif
    arg4 = 0.5*eta0
    dt2  = 2.*dt    ! [s]
    ! End specifying runtime parameters and constants
!-----------------------------------------------------------------
    ! Open, write to, and close logfile
    open(unit=llun,file=logfile,form="formatted",iostat=res)
    if(res /= 0) then                  ! An error has occured
      print*, "Error in opening logfile, status: ", res
      stop                             ! Stop the program
    endif
    write(unit=llun,FMT='(1X,A37,1X,4I6,1X,A7)',iostat=res) &
          'Flags : icase, ischeme, iev, ishap            : ', &
                                    icase,ischeme,iev,ishap,'-'
    if(res /= 0) then                  ! An error has occured
      print*, "Error in writing Flags to logfile, status: ", res
      stop                             ! Stop the program
    endif
    write(unit=llun,FMT='(1X,A37,1X,I20,1X,A7)',iostat=res) &
          '# of time steps to complete 1 cycle: ', Nfl,'-'
    if(res /= 0) then                  ! An error has occured
      print*, "Error in writing nfl to logfile, status: ", res
      stop                             ! Stop the program
    endif
    write(unit=llun,FMT='(1X,A37,1X,F20.6,1X,A7)',iostat=res) &
          'Length Lx of basin                  : ', Lx,'m'
    if(res /= 0) then                  ! An error has occured
      print*, "Error in writing Lx to logfile, status: ", res
```

```
   stop                              ! Stop the program
endif
write(unit=llun,FMT='(1X,A37,1X,F20.6,1X,A7)',iostat=res) &
      'Gravitational acceleration         : ', g,'m^2/s^2'
if(res /= 0) then                    ! An error has occured
   print*, "Error in writing g to logfile, status: ", res
   stop                              ! Stop the program
endif
write(unit=llun,FMT='(1X,A37,1X,F20.6,1X,A7)',iostat=res) &
      'Coriolis parameter f               : ', f,'m^2/s'
if(res /= 0) then                    ! An error has occured
   print*, "Error in writing f to logfile, status: ", res
   stop                              ! Stop the program
endif
write(unit=llun,FMT='(1X,A37,1X,F20.6,1X,A7)',iostat=res) &
      'Equilibrium depth H0               : ', H0,'m'
if(res /= 0) then                    ! An error has occured
   print*, "Error in writing H0 to logfile, status: ", res
   stop                              ! Stop the program
endif
write(unit=llun,FMT='(1X,A37,1X,2F20.6,1X,A7)',iostat=res) &
      'Space increments dx, dy                 : ',dx, dy, 'm'
if(res /= 0) then                    ! An error has occured
   print*, "Error in writing dx to logfile, status: ", res
   stop                              ! Stop the program
endif
write(unit=llun,FMT='(1X,A37,1X,F20.6,1X,A7)',iostat=res) &
      'Time step dt                       : ', dt,'s'
if(res /= 0) then                    ! An error has occured
   print*, "Error in writing dt to logfile, status: ", res
   stop                              ! Stop the program
endif
write(unit=llun,FMT='(1X,A37,1X,F20.6,1X,A7)',iostat=res) &
      'Deformation radius                 : ', Lr,'m'
```

```
  if(res /= 0) then                  ! An error has occured
     print*, "Error in writing Lr to logfile, status: ", res
     stop                            ! Stop the program
  endif
  write(unit=llun,FMT='(1X,A37,1X,F20.6,1X,A7)',iostat=res) &
        'Wave speed c0                      : ', c0,'m/s'
  if(res /= 0) then                  ! An error has occured
     print*, "Error in writing c0 to logfile, status: ", res
     stop                            ! Stop the program
  endif
  ! Close the logfile
  close(unit=llun)
  ! End opening, writing to, and closing the logfile
!-----------------------------------------------------------------
  ! Location of the auxiliary points along the x and y axes.
  ! Origo is the auxiliary point in grid cell (1,1).
  do j=1,jmax
     x(j) = (j-1)*dx
     if(icase == 1) then
        xkm(j) = x(j)    ! Case I dimension is only 10 [m]
     else
        xkm(j) = x(j)/1000.
     endif
  end do
  do k=1,kmax
     y(k) = (k-1)*dy
     ykm(k) = y(k)/1000.
  end do
!-----------------------------------------------------------------
  ! Relaxation parameter(s) needed for FRS boundary condition
  do j=1,jmax
     arg11 = (j-1)/3.
     arg12 = (jmax-j)/3.
     x(j)  = (j-1)*dx
```

```
      if(j <= Ng) then
         alf(j) = 1. - tanh(arg11)
      elseif(j >= jmax-Ng) then
         alf(j) = 1. - tanh(arg12)
      else
         alf(j) = 0.
      endif
   end do

! End front matter
!----------------------------------------------------------------
! Initialization
   ! Time step, time and time level counters
   ntime = 0
   time  = 0.
   htime = 0.
   dtime = 0.
   iold  = 1
   icur  = 2
   if(ischeme == 1) then
inew = 2                     ! FB-L has 2 time levels
   else
inew = 3
   endif
   ! Variables and equilibrium depth
   call init(U,V,h,eta,Up,Vp,hp,etap,Ue,Ve,etae,x,y,Hl,Hr, &
             jmax,kmax,nt,np,icase,arg4,H0,x0,y0, &
             dx,dy,Lr,c0,f,eta0)
   ! Save initial variables
   nsave = 1
   call savef(U,V,eta,xkm,Up,Vp,hp,etap,Ue,Ve,etae, &
              jmax,kmax,inew,nt,np,nsave,ishap,ibc,H0,res)

! End initialization
```

```
!-------------------------------------------------------------
  do ntime=1,ntmax                    ! Start the time loop
     print *, 'Working on time step no.: ', ntime
     ! New time in seconds, hours and days
     time  = time+dt
     htime = time/3600.
     dtime = htime/24.
     ! Update variables U,V,h,eta including the boundaries
     if(ischeme == 1) then            ! FB-L (linear)
        call fbl(U,V,h,eta,x,y,alf,Hl,Hr,jmax,kmax,iold,inew, &
                 nt,ntime,ibc,dx,dy,dt,f,g,H0,x0,y0)
     elseif(ischeme == 2) then        ! CTCS-L (linear)
        ! Euler step to avoid initial problem of CTCS
        if(ntime == 1) then
           call euler_l(U,V,h,eta,x,y,alf,Hl,Hr,jmax,kmax, &
                        iold,icur,nt,ntime,iev,ibc,dx,dy,dt, &
                        f,g,A,eta0,H0,x0,y0)
        else
           call ctcs_l(U,V,h,eta,x,y,alf,Hl,Hr,jmax,kmax, &
                       iold,icur,inew,nt,ntime,iev,ibc, &
                       dx,dy,dt,dt2,f,g,A,eta0,H0,x0,y0)
        endif
     else                             !  CTCS-N (non-linear)
        ! Euler step to avoid initial problem of CTCS
        if(ntime == 1) then
           call euler_n(U,V,h,eta,x,y,alf,Hl,Hr,jmax,kmax, &
                        iold,icur,nt,ntime,ibc,iev,dx,dy,dt, &
                        f,g,A,eta0,H0,x0,y0)
        else
           call ctcs_n(U,V,h,eta,x,y,alf,Hl,Hr,jmax,kmax, &
                       iold,icur,inew,nt,ntime,ibc,iev, &
                       dx,dy,dt,dt2,f,g,A,eta0,H0,x0,y0)
        endif
     endif
```

```
    ! Should we write to the Hovmoller file?
    if(mod(ntime,nhov) == 0) then
       write (unit =olunhov,fmt ='(1001F12.4)', iostat =res) &
          (eta(j,2,inew),j=1,jmax)
       if(res /= 0) then            ! An error has occured
          print*, 'Error saving Hovmoller-results.dat:'
          print*, 'status: ', res
          stop                      ! Stop the program
       endif
    endif
    ! Should we save the results?
    if( ntime == n2 ) then
       nsave = 2
       go to 13
    elseif( ntime == n3 ) then
       nsave = 3
       go to 13
    elseif( ntime == n4 ) then
       nsave = 4
       go to 13
    else
       go to 23
    endif
13  call savef(U,V,eta,xkm,Up,Vp,hp,etap,Ue,Ve,etae, &
               jmax,kmax,inew,nt,np,nsave,ishap,ibc,H0,res) &
    ! Swap indices
23  if(ischeme == 1) then           ! FB-L scheme
       isave = inew
       inew = iold
       iold = isave
    else
       if(ntime == 1) then          ! Euler scheme
          continue
       else                         ! CTCS-L & CTCS-N scheme
```

```
            isave = inew
            inew  = iold
            iold  = icur
            icur  = isave
         endif
      endif
  end do
! End of time loop
!-----------------------------------------------------------------
! Write results to files
  write(6,*) 'Writing results to files'
  call store(Ue,Ve,etae,Up,Vp,hp,etap,xkm,jmax,kmax,np,kp, &
             icase,olunh1,olunu1,olunv1,olunh2,olunu2,olunv2, &
             olunh3,olunu3,olunv3,olunh4,olunu4,olunv4, &
             olunhe,olunue,olunve,olunhp,olunup,olunvp,res)
! End writing to files
!-----------------------------------------------------------------
! End program in an orderly fashion
  ! Close output and log files
  close(unit=olunh1)
  close(unit=olunh2)
  close(unit=olunh3)
  close(unit=olunh4)
  close(unit=olunhe)
  close(unit=olunhp)
  close(unit=olunu1)
  close(unit=olunu2)
  close(unit=olunu3)
  close(unit=olunu4)
  close(unit=olunue)
  close(unit=olunup)
  close(unit=olunv1)
  close(unit=olunv2)
  close(unit=olunv3)
```

```
 close(unit=olunv4)
 close(unit=olunve)
 close(unit=olunvp)
 close(unit=olunhov)
2 format(3I4,1X,4F10.4)
 if( icase == 1 ) then
    write(6,*) 'Case I solved successfully'
 elseif( icase == 2) then
    write(6,*) 'Case II solved successfully'
 elseif( icase == 3) then
    write(6,*) 'Case III solved successfully'
 elseif( icase == 4) then
    write(6,*) 'Case IV solved successfully'
 endif
 if(ischeme == 1) then
    write(6,*) 'The Forward-Backward scheme (FB-L) was used'
 elseif( ischeme == 2 ) then
    if(iev == 0) then
       write(6,*) 'The CTCS-L scheme was employed. No EV added'
    elseif( iev == 1 )  then
       write(6,*) 'The CTCS-L scheme was employed. EV added'
    endif
 else
    if(iev == 0) then
       write(6,*) 'The CTCS-N scheme was employed. No EV added'
    elseif( iev == 1 )  then
       write(6,*) 'The CTCS-N scheme was employed. EV added'
    endif
 endif
 if( ishap == 1 ) then
    write(6,*) 'The Shapiro filter applied to results.'
 else
    write(6,*) 'No Shapiro filter applied.'
 endif
```

```
end program nonlinear_swe            ! End of main program
!-----------------------------------------------------------------
```

A.2 Subroutine `init.f90`

```
!////////////////////////////////////////////////////////////
!//                                                        //
!// Subroutine init                                        //
!//                                                        //
!// Called by main program                                 //
!//                                                        //
!// Purpose:                                               //
!//                                                        //
!// To initialize volume transports (U,V), water column    //
!// thickness (h), and sea surface deviation (eta).        //
!//                                                        //
!////////////////////////////////////////////////////////////

subroutine init(U,V,h,eta,Up,Vp,hp,etap, &
                Ue,Ve,etae,x,y,Hl,Hr, &
                jmax,kmax,nt,np,icase,arg4, &
                H0,x0,y0,dx,dy,Lr,c0,f,eta0)
  implicit none
  integer, intent(IN) :: jmax,kmax,nt,np,icase
  integer :: j,k,n,jp1,res
  real, intent(IN) :: H0,x0,y0,dx,dy,Lr,c0,f,eta0,arg4
  real, dimension(jmax,kmax,nt) :: U,V,h,eta
  real, dimension(jmax,kmax,np) :: Up,Vp,hp,etap
  real, dimension(jmax,kmax,np) :: Ue,Ve,etae
  real, dimension(jmax) :: x
  real, dimension(kmax) :: y,Hl,Hr
  real :: argx,argy,LL,DD
  real, allocatable, dimension(:) :: arg1 ! Initial y dependence
  real, allocatable, dimension(:) :: arg2 ! Initial x dependence
  real, allocatable, dimension(:) :: arg3 ! As arg1
  real, allocatable, dimension(:) :: arg5 ! As arg1
!------------------------------------------------------------
```

```
  ! Allocate allocatable variables
  allocate (arg1(kmax), STAT=res)
  if(res /= 0) then
     print *, 'Error in allocating space for arg1, status:', res
     stop
  endif
  allocate (arg2(jmax), STAT=res)
  if(res /= 0) then
     print *, 'Error in allocating space for arg2, status:', res
     stop
  endif
  allocate (arg3(kmax), STAT=res)
  if(res /= 0) then
     print *, 'Error in allocating space for arg3, status:', res
     stop
  endif
  allocate (arg5(kmax), STAT=res)
  if(res /= 0) then
     print *, 'Error in allocating space for arg5, status:', res
     stop
  endif
! End allocating variables
!-----------------------------------------------------------------
! Give a default value (a funny one though) to all variables in
! all grid cells
  ! Variables U, V, h and eta at all time levels
  do n = 1,nt
     do k=1,kmax
        do j=1,jmax
           U(j,k,n)   = 999.
           V(j,k,n)   = 999.
           h(j,k,n)   = 999.
           eta(j,k,n) = 999.
        end do
```

```
      end do
   end do

   ! Variables Up, Vp. hp, etap, Ue, Ve, etae to be saved and
   ! written to file np times
   do n = 1,np
      do k=1,kmax
         do j=1,jmax
            Up(j,k,n)   = 999.
            Vp(j,k,n)   = 999.
            hp(j,k,n)   = 999.
            etap(j,k,n) = 999.
            Ue(j,k,n)   = 999.
            Ve(j,k,n)   = 999.
            etae(j,k,n) = 999.
         end do
      end do
   end do
! End giving all variables a default value
!-----------------------------------------------------------------
! Assign initial values to variables for each case
   do n=1,nt
      if( icase == 1 ) then                  ! Case A: Abrupt step
         ! h and eta
         do k=1,kmax-1
            do j=1,jmax
               if(x(j) < x0) then
                  eta(j,k,n) = eta0
               elseif(x(j) == x0) then
                  eta(j,k,n) = 0.
               else
                  eta(j,k,n) = - eta0
               endif
               h(j,k,n) = eta(j,k,n) + H0
```

```
      end do
      Hl(k) = h(1,k,n)
      Hr(k) = h(jmax,k,n)
   end do
   ! U and V
   do k=1,kmax
      do j=1,jmax
         U(j,k,n)   = 0.
         V(j,k,n)   = 0.
      end do
   end do
elseif( icase == 2 ) then            ! Case B
   ! Compute some helpful constants
   LL = 15.*dx
   DD = 50.*dx
   ! eta and h
   do k=1,kmax
      argy = y(k) - 0.5*dy - y0
      do j=1,jmax
         argx = x(j) - 0.5*dx - x0
         eta(j,k,n) = arg4*( 1. + &
             tanh((DD - sqrt( argx*argx + argy*argy) )/LL))
         h(j,k,n)   = eta(j,k,n) + H0
         U(j,k,n)   = 0.
         V(j,k,n)   = 0.
      end do
      Hl(k) = h(1,k,n)
      Hr(k) = h(jmax,k,n)
   end do
else                                  ! Cases C and D
   ! Compute some helpful arrays
   do k=1,kmax
      argy    = y(k) - 0.5*dy - y0
      arg1(k) = - (1./Lr)*sqrt( argy**2. )
```

```
   arg3(k) = 0.5*c0*sign( 1.,argy )
   arg5(k) = arg4*exp( arg1(k) )
end do
do j=1,jmax
   argx    = x(j) - 0.5*dx - x0
   arg2(j) = tanh( (3./Lr)*( Lr - sqrt( argx*argx ) ) )
end do
! eta and h
do k=2,kmax-1
   do j=2,jmax
      eta(j,k,n) = arg5(k)*( 1. + arg2(j) )
      h(j,k,n)   = eta(j,k,n) + H0
   end do
   ! Cyclic boundary condition
   eta(1,k,n) = eta(jmax,k,n)
   h(1,k,n)   = h(jmax,k,n)
end do
! U
do k=2,kmax-1
   do j=2,jmax
      if(j == jmax) then ! Cyclic boundary condition
         jp1 = 2
      else
         jp1 = j+1
      endif
      U(j,k,n) = arg3(k)*( eta(jp1,k,n) + eta(j,k,n) )
   end do
   ! Cyclic boundary condition
   U(1,k,n) = U(jmax,k,n)
end do
! Closed boundary condition U
do j=1,jmax
   U(j,1,n)    = - U(j,2,n)
   U(j,kmax,n) = - U(j,kmax-1,n)
```

```
        end do
        ! V including boundaries
        do k=1,kmax-1
           do j=1,jmax
              V(j,k,n) = 0.
           end do
        end do
     endif
  end do
! End assigning real values to variables for all cases
!-----------------------------------------------------------------

  return

end subroutine init
```

A.3 Subroutine `savef.f90`

```
!////////////////////////////////////////////////////////////
!//                                                        //
!// Subroutine savef                                       //
!//                                                        //
!// Called by main program: nonlinear_swe                  //
!//                                                        //
!// Purpose:                                               //
!//                                                        //
!// To save the variables for later storing at their       //
!// respective grid points (etae, Ue and Ve)               //
!//                                                        //
!////////////////////////////////////////////////////////////



subroutine savef(U,V,eta,xkm,Up,Vp,hp,etap,Ue,Ve,etae, &
jmax,kmax,inew,nt,np,nsave,ishap,ibc,H0,res)
  implicit none
  integer, intent(in)            :: jmax,kmax
  integer, intent(in)            :: inew,nt,np,nsave,ishap
  integer                        :: j,k,res,jp1,ibc
  real, dimension (jmax,kmax,nt) :: U,V,h,eta
  real, dimension(jmax)          :: xkm
  real, dimension(jmax,kmax,np)  :: Up,Vp,hp,etap,Ue,Ve,etae
  real :: H0

!------------------------------------------------------------
! Save variables
  if(ishap == 0) then               ! Raw model results
     do k=2,kmax-1
        do j=1,jmax
           etae(j,k,nsave) = eta(j,k,inew)
```

```
      end do
   end do
   do k=1,kmax-1
      do j=1,jmax
         Ve(j,k,nsave) = V(j,k,inew)
      end do
   end do
   do k=2,kmax-1
      do j=1,jmax
         Ue(j,k,nsave) = U(j,k,inew)
      end do
   end do
else                        ! ishap = 1 => Apply Shapiro filter
   if(ibc == 1) then               ! Cyclic boundary condition
      do k=3,kmax-2
         do j=2,jmax
            if(j == jmax) then
               jp1 = 2
            else
               jp1=j+1
            endif
            etae(j,k,nsave) = 0.25*( eta(jp1,k,inew) + &
                                     eta(j-1,k,inew) + &
                                     eta(j,k+1,inew) + &
                                     eta(j,k-1,inew) )
            Ue(j,k,nsave)   = 0.25*( U(jp1,k,inew) + &
                                     U(j-1,k,inew) + &
                                     U(j,k+1,inew) + &
                                     U(j,k-1,inew) )
            Ve(j,k,nsave)   = 0.25*( V(jp1,k,inew) + &
                                     V(j-1,k,inew) + &
                                     V(j,k+1,inew) + &
                                     V(j,k-1,inew) )
         end do
```

```
        etae(1,k,nsave) = etae(jmax,k,nsave) ! j=1
        Ue(1,k,nsave)   = Ue(jmax,k,nsave)
        Ve(1,k,nsave)   = Ve(jmax,k,nsave)
end do
do j=2,jmax                                  ! k=2, k=kmax-1
        if(j == jmax) then
           jp1 = 2
        else
           jp1=j+1
        endif
        etae(j,2,nsave) = eta(j,2,inew) + &
                  0.25*( eta(jp1,2,inew) - &
                      2.*eta(j,2,inew) + &
                         eta(j-1,2,inew) )
        Ue(j,2,nsave)   = U(j,2,inew) + &
                  0.25*( U(jp1,2,inew) - &
                      2.*U(j,2,inew) + &
                         U(j-1,2,inew) )
        Ve(j,2,nsave)   = V(j,k,inew) + &
                  0.25*( V(jp1,k,inew) - &
                      2.*V(j,k,inew) + &
                         V(j-1,k,inew) )
        Ve(j,1,nsave)   = 0.
        etae(j,kmax-1,nsave) = eta(j,kmax-1,inew) + &
                        0.25*( eta(jp1,kmax-1,inew) - &
                            2.*eta(j,kmax-1,inew) + &
                               eta(j-1,kmax-1,inew) )
        Ue(j,kmax-1,nsave)   = U(j,kmax-1,inew) + &
                        0.25*( U(jp1,kmax-1,inew) - &
                            2.*U(j,kmax-1,inew) + &
                               U(j-1,kmax-1,inew) )
        Ve(j,kmax-1,nsave) = 0.
end do
     ! j=1, k=2
```

```
etae(1,2,nsave)      = etae(jmax,2,nsave)
Ue(1,2,nsave)        = Ue(jmax,2,nsave)
Ve(1,2,nsave)        = Ve(jmax,2,nsave)
     ! j=1, k=kmax-
etae(1,kmax-1,nsave) = etae(jmax,kmax-1,nsave) 1
Ue(1,kmax-1,nsave)   = Ue(jmax,kmax-1,nsave)
Ve(1,kmax-1,nsave)   = 0.
  else                              ! Flow Relaxation Scheme
     do k=3,kmax-2
        do j=2,jmax-1
           jp1=j+1
           etae(j,k,nsave) = 0.25*( eta(jp1,k,inew) + &
                                    eta(j-1,k,inew) + &
                                    eta(j,k+1,inew) + &
                                    eta(j,k-1,inew) )
           Ue(j,k,nsave)   = 0.25*( U(jp1,k,inew) + &
                                    U(j-1,k,inew) + &
                                    U(j,k+1,inew) + &
                                    U(j,k-1,inew) )
           Ve(j,k,nsave)   = 0.25*( V(jp1,k,inew) + &
                                    V(j-1,k,inew) + &
                                    V(j,k+1,inew) + &
                                    V(j,k-1,inew) )
        end do
        etae(1,k,nsave) = eta(1,k,inew) ! j=1
        Ue(1,k,nsave)   = U(1,k,inew)
        Ve(1,k,nsave)   = V(1,k,inew)
        etae(jmax,k,nsave) = eta(jmax,k,inew) ! j=jmax
        Ue(jmax,k,nsave)   = U(jmax,k,inew)
        Ve(jmax,k,nsave)   = V(jmax,k,inew)
     end do
do j=2,jmax-1                       ! k=1,2 and k=kmax-1
        jp1=j+1
        etae(j,2,nsave) = eta(j,2,inew) + &
```

```
                0.25*( eta(jp1,2,inew) - &
                    2.*eta(j,2,inew) + &
                       eta(j-1,2,inew) )
      Ue(j,2,nsave)   = U(j,2,inew) + &
                0.25*( U(jp1,2,inew) - &
                    2.*U(j,2,inew) + &
                       U(j-1,2,inew) )
      Ve(j,2,nsave)   = V(j,k,inew) + &
                0.25*( V(jp1,k,inew) - &
                    2.*V(j,k,inew) + &
                       V(j-1,k,inew) )
      Ve(j,1,nsave)   = 0.
      etae(j,kmax-1,nsave) = eta(j,kmax-1,inew) + &
                     0.25*( eta(jp1,kmax-1,inew) - &
                         2.*eta(j,kmax-1,inew) + &
                            eta(j-1,kmax-1,inew) )
      Ue(j,kmax-1,nsave)   = U(j,kmax-1,inew) + &
                     0.25*( U(jp1,kmax-1,inew) - &
                         2.*U(j,kmax-1,inew) + &
                            U(j-1,kmax-1,inew) )
      Ve(j,kmax-1,nsave) = 0.
   end do
   etae(1,2,nsave)      = eta(1,2,inew) ! j=1, k=2
   Ue(1,2,nsave)        = U(1,2,inew)
   Ve(1,2,nsave)        = V(1,2,inew)
   etae(1,kmax-1,nsave) = eta(1,kmax-1,inew)
   Ue(1,kmax-1,nsave)   = U(1,kmax-1,inew)
   Ve(1,kmax-1,nsave)   = 0.
   etae(jmax,2,nsave)      = eta(jmax,2,inew) ! j=jmax, k=2
   Ue(jmax,2,nsave)        = U(jmax,2,inew)
   Ve(jmax,2,nsave)        = V(jmax,2,inew)
   etae(jmax,kmax-1,nsave) = eta(jmax,kmax-1,inew)
   Ue(jmax,kmax-1,nsave)   = U(jmax,kmax-1,inew)
   Ve(jmax,kmax-1,nsave)   = 0.
```

```
      endif
    endif

! End saving variables
!-----------------------------------------------------------------

    return

end subroutine savef
```

A.4 Subroutine `store.f90`

```
!////////////////////////////////////////////////////////////
!//                                                        //
!// Subroutine store                                       //
!//                                                        //
!// Called by main program: nonlinear_swe                  //
!//                                                        //
!// Purpose:                                               //
!//                                                        //
!// To store the results to be plotted                     //
!//                                                        //
!////////////////////////////////////////////////////////////

subroutine store(Ue,Ve,etae,Up,Vp,hp,etap,xkm,jmax,kmax,np, &
                 kp,icase,olunh1,olunu1,olunv1, &
                 olunh2,olunu2,olunv2,olunh3,olunu3,olunv3, &
                 olunh4,olunu4,olunv4,olunhe,olunue,olunve, &
                 olunhp,olunup,olunvp,res)
  implicit none
  integer, intent(in)           :: jmax,kmax,np,kp,icase
  integer, intent(in)           :: olunhp,olunup,olunvp
  integer, intent(in)           :: olunhe,olunue,olunve
  integer, intent(in)           :: olunh1,olunu1,olunv1
  integer, intent(in)           :: olunh2,olunu2,olunv2
  integer, intent(in)           :: olunh3,olunu3,olunv3
  integer, intent(in)           :: olunh4,olunu4,olunv4
  integer                       :: j,k,n,res
  real, dimension(jmax)         :: xkm
  real, dimension(jmax,kmax)    :: U,V,h,eta
  real, dimension(jmax,kmax,np) :: Ue,Ve,etae,Up,Vp,hp,etap
!-------------------------------------------------------------
  ! Write to files
  if(icase == 1) then ! Case I: Breaking of a dam. No rotation
```

```
do k=2,kmax-1
   ! eta: etae fields
   write (unit =olunh1, fmt ='(501F12.7)', iostat =res) &
                  (etae(j,k,1),j=1,jmax)
   if(res /= 0) then                 ! An error has occured
      print *, &
      'Error saving to initial_eta-results, status:', res
      stop                           ! stop the program
   endif
   write (unit =olunh2, fmt ='(501F12.7)', iostat =res) &
                  (etae(j,k,2),j=1,jmax)
   if(res /= 0) then                 ! An error has occured
      print *, &
      'Error saving to 2nd_eta-results, status:', res
      stop                           ! stop the program
   endif
   write (unit =olunh3, fmt ='(501F12.7)', iostat =res) &
                  (etae(j,k,3),j=1,jmax)
   if(res /= 0) then                 ! An error has occured
      print *, &
      'Error saving to 3rd_eta-results, status:', res
      stop                           ! stop the program
   endif
   write (unit =olunh4, fmt ='(501F12.7)', iostat =res) &
                  (etae(j,k,4),j=1,jmax)
   if(res /= 0) then                 ! An error has occured
      print *, &
      'Error saving to 4th_eta-results, status:', res
      stop                           ! stop the program
   endif
   ! U: Ue fields
   write (unit =olunu1, fmt ='(501F12.7)', iostat =res) &
                  (Ue(j,k,1),j=1,jmax)
   if(res /= 0) then                 ! An error has occured
```

```
   print *, &
   'Error saving to initial_U-results, status:', res
   stop                          ! stop the program
endif
write (unit =olunu2, fmt ='(501F12.7)', iostat =res) &
               (Ue(j,k,2),j=1,jmax)
if(res /= 0) then                ! An error has occured
   print *, &
   'Error saving to 2nd_U-results, status:', res
   stop                          ! stop the program
endif
write (unit =olunu3, fmt ='(501F12.7)', iostat =res) &
               (Ue(j,k,3),j=1,jmax)
if(res /= 0) then                ! An error has occured
   print *, &
   'Error saving to 3rd_U-results, status:', res
   stop                          ! stop the program
endif
write (unit =olunu4, fmt ='(501F12.7)', iostat =res) &
               (Ue(j,k,4),j=1,jmax)
if(res /= 0) then                ! An error has occured
   print *, &
   'Error saving to 4th_U-results, status:', res
   stop                          ! stop the program
endif
! V: Ve field
write (unit =olunv1, fmt ='(501F12.7)', iostat =res) &
               (Ve(j,k,1),j=1,jmax)
if(res /= 0) then                ! An error has occured
   print *, &
   'Error saving to initial_V-results, status:', res
   stop                          ! stop the program
endif
write (unit =olunv2, fmt ='(501F12.7)', iostat =res) &
```

```
                (Ve(j,k,2),j=1,jmax)
      if(res /= 0) then                ! An error has occured
         print *, &
         'Error saving to 2nd_V-results, status:', res
         stop                          ! stop the program
      endif
      write (unit =olunv3, fmt ='(501F12.7)', iostat =res) &
                (Ve(j,k,3),j=1,jmax)
      if(res /= 0) then                ! An error has occured
         print *, &
         'Error saving to 3rd_V-results, status:', res
         stop                          ! stop the program
      endif
      write (unit =olunv4, fmt ='(501F12.7)', iostat =res) &
                (Ve(j,k,4),j=1,jmax)
      if(res /= 0) then                ! An error has occured
         print *, &
         'Error saving to 4th_V-results, status:', res
         stop                          ! stop the program
      endif
   end do
elseif (icase == 2) then   ! Case II: Adjustment under gravity
   do k=2,kmax-1
 ! eta: etae fields
      write (unit =olunh1, fmt ='(801F12.7)', iostat =res) &
                (etae(j,k,1),j=1,jmax)
      if(res /= 0) then                ! An error has occured
         print *, &
         'Error saving to initial_eta-results, status:', res
         stop                          ! stop the program
      endif
      write (unit =olunh2, fmt ='(801F12.7)', iostat =res) &
                (etae(j,k,2),j=1,jmax)
      if(res /= 0) then                ! An error has occured
```

```
   print *, &
   'Error saving to 2nd_eta-results, status:', res
   stop                          ! stop the program
endif
write (unit =olunh3, fmt ='(801F12.7)', iostat =res) &
               (etae(j,k,3),j=1,jmax)
if(res /= 0) then                ! An error has occured
   print *, &
   'Error saving to 3rd_eta-results, status:', res
   stop                          ! stop the program
endif
write (unit =olunh4, fmt ='(801F12.7)', iostat =res) &
               (etae(j,k,4),j=1,jmax)
if(res /= 0) then                ! An error has occured
   print *, &
   'Error saving to 4th_eta-results, status:', res
   stop                          ! stop the program
endif
! U: Ue fields
write (unit =olunu1, fmt ='(801F12.7)', iostat =res) &
               (Ue(j,k,1),j=1,jmax)
if(res /= 0) then                ! An error has occured
   print *, &
   'Error saving to initial_U-results, status:', res
   stop                          ! stop the program
endif
write (unit =olunu2, fmt ='(801F12.7)', iostat =res) &
               (Ue(j,k,2),j=1,jmax)
if(res /= 0) then                ! An error has occured
   print *, &
   'Error saving to 2nd_U-results, status:', res
   stop                          ! stop the program
endif
write (unit =olunu3, fmt ='(801F12.7)', iostat =res) &
```

```
              (Ue(j,k,3),j=1,jmax)
if(res /= 0) then                ! An error has occured
   print *, &
   'Error saving to 3rd_U-results, status:', res
   stop                          ! stop the program
endif
write (unit =olunu4, fmt ='(801F12.7)', iostat =res) &
              (Ue(j,k,4),j=1,jmax)
if(res /= 0) then                ! An error has occured
   print *, &
   'Error saving to 4th_U-results, status:', res
   stop                          ! stop the program
endif
! V: Ve field
write (unit =olunv1, fmt ='(801F12.7)', iostat =res) &
              (Ve(j,k,1),j=1,jmax)
if(res /= 0) then                ! An error has occured
   print *, &
   'Error saving to initial_V-results, status:', res
   stop                          ! stop the program
endif
write (unit =olunv2, fmt ='(801F12.7)', iostat =res) &
              (Ve(j,k,2),j=1,jmax)
if(res /= 0) then                ! An error has occured
   print *, &
   'Error saving to 2nd_V-results, status:', res
   stop                          ! stop the program
endif
write (unit =olunv3, fmt ='(801F12.7)', iostat =res) &
              (Ve(j,k,3),j=1,jmax)
if(res /= 0) then                ! An error has occured
   print *, &
   'Error saving to 3rd_V-results, status:', res
   stop                          ! stop the program
```

```
      endif
      write (unit =olunv4, fmt ='(801F12.7)', iostat =res) &
                    (Ve(j,k,4),j=1,jmax)
      if(res /= 0) then                 ! An error has occured
         print *, &
         'Error saving to 4th_V-results, status:', res
         stop                           ! stop the program
      endif
   end do
else ! Case III/IV: Small/large Kelvin waves
   do k=2,kmax-1
 ! eta: etae fields
      write (unit =olunh1, fmt ='(1001F12.7)', iostat =res) &
                    (etae(j,k,1),j=1,jmax)
      if(res /= 0) then                 ! An error has occured
         print *, &
         'Error saving to initial_eta-results, status:', res
         stop                           ! stop the program
      endif
      write (unit =olunh2, fmt ='(1001F12.7)', iostat =res) &
                    (etae(j,k,2),j=1,jmax)
      if(res /= 0) then                 ! An error has occured
         print *, &
         'Error saving to 2nd_eta-results, status:', res
         stop                           ! stop the program
      endif
      write (unit =olunh3, fmt ='(1001F12.7)', iostat =res) &
                    (etae(j,k,3),j=1,jmax)
      if(res /= 0) then                 ! An error has occured
         print *, &
         'Error saving to 3rd_eta-results, status:', res
         stop                           ! stop the program
      endif
      write (unit =olunh4, fmt ='(1001F12.7)', iostat =res) &
```

```
                (etae(j,k,4),j=1,jmax)
if(res /= 0) then                  ! An error has occured
   print *, &
   'Error saving to 4th_eta-results, status:', res
   stop                            ! stop the program
endif
! U: Ue fields
write (unit =olunu1, fmt ='(1001F12.7)', iostat =res) &
                (Ue(j,k,1),j=1,jmax)
if(res /= 0) then                  ! An error has occured
   print *, &
   'Error saving to initial_U-results, status:', res
   stop                            ! stop the program
endif
write (unit =olunu2, fmt ='(1001F12.7)', iostat =res) &
                (Ue(j,k,2),j=1,jmax)
if(res /= 0) then                  ! An error has occured
   print *, &
   'Error saving to 2nd_U-results, status:', res
   stop                            ! stop the program
endif
write (unit =olunu3, fmt ='(1001F12.7)', iostat =res) &
                (Ue(j,k,3),j=1,jmax)
if(res /= 0) then                  ! An error has occured
   print *, &
   'Error saving to 3rd_U-results, status:', res
   stop                            ! stop the program
endif
write (unit =olunu4, fmt ='(1001F12.7)', iostat =res) &
                (Ue(j,k,4),j=1,jmax)
if(res /= 0) then                  ! An error has occured
   print *, &
   'Error saving to 4th_U-results, status:', res
   stop                            ! stop the program
```

```
      endif
      ! V: Ve field
      write (unit =olunv1, fmt ='(1001F12.7)', iostat =res) &
                    (Ve(j,k,1),j=1,jmax)
      if(res /= 0) then                 ! An error has occured
         print *, &
         'Error saving to initial_V-results, status:', res
         stop                           ! stop the program
      endif
      write (unit =olunv2, fmt ='(1001F12.7)', iostat =res) &
                    (Ve(j,k,2),j=1,jmax)
      if(res /= 0) then                 ! An error has occured
         print *, &
         'Error saving to 2nd_V-results, status:', res
         stop                           ! stop the program
      endif
      write (unit =olunv3, fmt ='(1001F12.7)', iostat =res) &
                    (Ve(j,k,3),j=1,jmax)
      if(res /= 0) then                 ! An error has occured
         print *, &
         'Error saving to 3rd_V-results, status:', res
         stop                           ! stop the program
      endif
      write (unit =olunv4, fmt ='(1001F12.7)', iostat =res) &
                    (Ve(j,k,4),j=1,jmax)
      if(res /= 0) then                 ! An error has occured
         print *, &
         'Error saving to 4th_V-results, status:', res
         stop                           ! stop the program
      endif
   end do
 endif
!
! eta: Raw data along southern boundary (k=kp) at grid
```

```
  ! points (etae) and at auxiliary points (etap)
  do j=1,jmax
     write (unit =olunhe, fmt ='(5F12.7)', iostat =res) xkm(j) &
,etae(j,kp,1), etae(j,kp,2), etae(j,kp,3), etae(j,kp,4)
     if(res /= 0) then                    ! An error has occured
        print *, &
        'Error saving to file etae-results, status:', res
        stop ! stop the program
     endif
     write (unit =olunhe, fmt ='(5F12.7)', iostat =res) xkm(j) &
,etap(j,kp,1), etap(j,kp,2), etap(j,kp,3), etap(j,kp,4)
     if(res /= 0) then                    ! An error has occured
        print *, &
        'Error saving to file etap-results, status:', res
        stop ! stop the program
     endif
  end do
  ! U: Raw data along southern boundary (k=kp) at grid
  ! points (Ue) and at auxiliary points (Up)
  do j=1,jmax
     write (unit =olunue, fmt ='(5F12.7)', iostat =res) xkm(j) &
,Ue(j,kp,1), Ue(j,kp,2), Ue(j,kp,3), Ue(j,kp,4)
     if(res /= 0) then                    ! An error has occured
        print *, &
        'Error saving to file Ue-results, status:', res
        stop ! stop the program
     endif
     write (unit =olunup, fmt ='(5F12.7)', iostat =res) xkm(j) &
,Up(j,kp,1), Up(j,kp,2), Up(j,kp,3), Up(j,kp,4)
     if(res /= 0) then                    ! An error has occured
        print *, &
        'Error saving to file Up-results, status:', res
        stop ! stop the program
     endif
```

```
  end do

  ! V: Raw data along southern boundary (k=kp) at grid
  !points (Ve) and at auxiliary points (Vp)
  do j = 1,jmax
     write (unit =olunve, fmt ='(5F12.7)', iostat =res) xkm(j) &
,Ve(j,kp,1), Ve(j,kp,2), Ve(j,kp,3), Ve(j,kp,4)
     if(res /= 0) then                    ! An error has occured
        print *, &
        'Error saving to file Ve-results, status:', res
        stop           ! stop the program
     endif
     write (unit =olunvp, fmt ='(5F12.7)', iostat =res) xkm(j) &
,Vp(j,kp,1), Vp(j,kp,2), Vp(j,kp,3), Vp(j,kp,4)
     if(res /= 0) then                    ! An error has occured
        print *, &
        'Error saving to file Vp-results, status:', res
        stop           ! stop the program
     endif
  end do

  return

end subroutine store
```

A.5 Subroutine ctcs-n.f90

```
!////////////////////////////////////////////////////////////
!//                                                        //
!// Subroutine ctcs_n                                      //
!//                                                        //
!// Called by main program: nonlinear_swe                  //
!//                                                        //
!// Purpose:                                               //
!//                                                        //
!// To update thickness and volume transports using the    //
!// nonlinear CTCS scheme.                                 //
!// First step (ntime=1) uses subroutine Euler             //
!//                                                        //
!////////////////////////////////////////////////////////////

subroutine ctcs_n(U,V,h,eta,x,y,alf,Hl,Hr,jmax,kmax,iold,icur, &
                  inew,nt,ntime,ibc,iev,dx,dy,dt,dt2, &
                  f,g,A,eta0,H0,x0,y0)
  implicit none
!------------------------------------------------------------
! Front matter
  ! Declarations
  integer, intent(in)           :: jmax,kmax,iold,icur,inew
  integer, intent(in)           :: nt,ntime,ibc,iev
  real, intent(in)              :: dx,dy,dt,dt2,f,g,A,eta0,H0
  real, intent(in)              :: x0,y0
  integer                       :: j,k,n,jp1
  real, dimension(jmax,kmax,nt) :: U,V,h,eta
  real, dimension(kmax)         :: corV,Hl,Hr,y
  real, dimension(jmax)         :: alf,x
  real                          :: CU,CV,PU,PV,MU,MV
  real                          :: ax,ay,pru,prv,a4x,a4y
  real                          :: mixu,mixv,R,corU
```

```
  real                            :: AUx,AUy,AVx,AVy
  real                            :: Uhjpk,Uhjk,UAjk,UAjkm
  real                            :: VAjk,VAjkm,hAjk,hAjkm
  real                            :: UAjmk,VAjmk,Vhjk,Vhjkm,hAjmk
!-----------------------------------------------------------------
  ! Compute constants needed
  ax   = dt2/dx
  ay   = dt2/dy
  pru  = 0.5*g*ax
  prv  = 0.5*g*ay
  mixu = iev*dt2*A/(dx*dx)
  mixv = iev*dt2*A/(dy*dy)
  R    = 1./( 1. + mixu + mixv )
  a4x  = 0.25*ax
  a4y  = 0.25*ay
  corU = 0.5*f*dt

  do k=1,kmax
corV(k) = 0.5*f*dt
  end do
  corV(2)      = f*dt
  corV(kmax-1) = f*dt
  corV(kmax)   = f*dt
! End front matter
!-----------------------------------------------------------------
! Update variables
  ! Recall the cyclic boundary condition east-west.
  ! So when j=jmax all references to j+1 must be replaced by 2
  !---------------------------------------------------------------
  ! 1: h and eta
  do k=2,kmax-1
     do j=2,jmax
        ! Update h
        h(j,k,inew) = h(j,k,iold) - &
```

```
                    ax*( U(j,k,icur) - U(j-1,k,icur) ) - &
                    ay*( V(j,k,icur) - V(j,k-1,icur) )
      ! Update eta
      eta(j,k,inew) = h(j,k,inew) - H0
   end do
   if(ibc == 1) then                ! Cyclic boundary condition
      h(1,k,inew) = h(jmax,k,inew)
      eta(1,k,inew) = eta(jmax,k,inew)
   else
      do j=1,jmax                   ! FRS boundary condition
         if(x(j) <= x0) then
            h(j,k,inew) = ( 1. - alf(j) )*h(j,k,inew) + &
                                  alf(j)*Hl(k)
         else
            h(j,k,inew)   = ( 1. - alf(j) )*h(j,k,inew) + &
                                  alf(j)*Hr(k)
         endif
         eta(j,k,inew)   = h(j,k,inew) - H0
      end do
   endif
end do
!-------------------------------------------------------------
! 2: U
do k=2,kmax-1
   if(ibc == 1) then                ! Cyclic boundary condition
      do j=2,jmax
         ! Recall cyclic boundary condition east-west => all
         ! references to j+1 must be replaced by 2 when
         ! j = jmax
         if(j == jmax) then
            jp1 = 2
         else
            jp1 = j+1
         endif
```

```
! Coriolis term U
CU = corV(k)*( V(j,k-1,icur) + V(j,k,icur) + &
               V(jp1,k,icur) + V(jp1,k-1,icur) )
! Pressure term U
PU = - pru*( h(jp1,k,icur) + h(j,k,icur) )* &
           ( eta(jp1,k,icur) - eta(j,k,icur) )
! Advective, nonlinear flux terms U
Uhjpk  = U(jp1,k  ,icur) + U(j  ,k  ,icur)
Uhjk   = U(j  ,k  ,icur) + U(j-1,k  ,icur)
UAjk   = U(j  ,k+1,icur) + U(j  ,k  ,icur)
UAjkm  = U(j  ,k  ,icur) + U(j  ,k-1,icur)
VAjk   = V(jp1,k  ,icur) + V(j  ,k  ,icur)
VAjkm  = V(jp1,k-1,icur) + V(j  ,k-1,icur)
if(k == 2) then
   hAjkm = 2.*( h(j,k,icur) + h(jp1,k,icur) )
else
   hAjkm = h(j,k-1,icur) + h(j,k,icur) + &
           h(jp1,k,icur) + h(jp1,k-1,icur)
endif
if(k == kmax-1) then
   hAjk  = 2.*( h(j,k,icur) + h(jp1,k,icur) )
else
   hAjk  = h(j,k,icur) + h(j,k+1,icur) + &
           h(jp1,k+1,icur) + h(jp1,k,icur)
endif
AUx  = - a4x*( ( (Uhjpk*Uhjpk)/h(jp1,k,icur) ) - &
               ( ( Uhjk*Uhjk  )/h(j,k,icur) ) )
AUy  = -  ay*( ( (UAjk*VAjk  )/hAjk ) - &
               ( ( UAjkm*VAjkm )/hAjkm ) )
! Mixing terms (Dufort-Frankel) U
MU   = mixu*( U(jp1,k,icur) - U(j,k,iold) + &
                              U(j-1,k,icur) ) &
     + mixv*( U(j,k+1,icur) - U(j,k,iold) + &
                              U(j,k-1,icur) )
```

```
         ! Update U
         U(j,k,inew) = ( U(j,k,iold) + CU + PU + &
                                       AUx + AUy + MU )*R
      end do
   else                                ! FRS boundary condition
      do j=1,jmax
         U(j,k,inew)   = (1. - alf(j))*U(j,k,inew) + alf(j)*0.
      end do
   endif
end do
! Closed boundary condition U
do j=1,jmax
   U(j,1,inew)    = - U(j,2,inew)
   U(j,kmax,inew) = - U(j,kmax-1,inew)
end do
!-------------------------------------------------------------
! 3: V
corU = 0.5*f*dt
do k=2,kmax-2
   if(ibc == 1) then               ! Cyclic boundary condition
      do j=2,jmax
         ! Recall cyclic boundary condition east-west => all
         ! references to j+1 must be replaced by 2 when
         ! j = jmax
         if(j == jmax) then
            jp1 = 2
         else
            jp1 = j+1
         endif
         ! Coriolis term V
         CV    = - corU*( U(j-1,k,icur) + U(j-1,k+1,icur) + &
                          U(j,k+1,icur) + U(j,k,icur) )
         ! Pressure term V
         PV    = - prv*( h(j,k+1,icur) + h(j,k,icur) )* &
```

```
                    ( eta(j,k+1,icur) - eta(j,k,icur) )
          ! The advective, non-linear flux terms V
          UAjk  = U(j  ,k+1,icur) + U(j  ,k  ,icur)
          UAjmk = U(j-1,k+1,icur) + U(j-1,k  ,icur)
          VAjk  = V(jp1,k  ,icur) + V(j  ,k  ,icur)
          VAjmk = V(j  ,k  ,icur) + V(j-1,k  ,icur)
          Vhjk  = V(j  ,k+1,icur) + V(j  ,k  ,icur)
          Vhjkm = V(j  ,k  ,icur) + V(j  ,k-1,icur)
          hAjk  = h(j,k  ,icur) + h(j,k+1,icur) + &
                  h(jp1,k+1,icur) + h(jp1,k,icur)
          hAjmk = h(j-1,k,icur) + h(j-1,k+1,icur) + &
                  h(j,k+1,icur) + h(j,k,icur)
          AVx   = - ax*( UAjk*VAjk/hAjk - UAjmk*VAjmk/hAjmk )
          AVy   = - a4y*( Vhjk*Vhjk/h(j,k+1,icur) - &
                          Vhjkm*Vhjkm/h(j,k,icur) )
          ! Mixing terms (Dufort-Frankel) V
          MV    = mixu*( V(jp1,k,icur) - V(j,k,iold) + &
                                         V(j-1,k,icur) ) &
                + mixv*( V(j,k+1,icur) - V(j,k,iold) + &
                                         V(j,k-1,icur) )
          ! Update V
          V(j,k,inew) = ( V(j,k,iold) + CV + PV + &
                                         AVx + AVy + MV )*R
end do
     else
        do j=1,jmax                      ! FRS boundary condition
          V(j,k,inew)   = (1. - alf(j))*V(j,k,inew) + alf(j)*0.
        end do
     endif
  end do
  ! Closed boundary condition V
  do j=1,jmax
V(j,1,inew) = 0.
V(j,kmax-1,inew) = 0.
```

```
  end do
! End updating variables
!-----------------------------------------------------------------

  return

end subroutine ctcs_n
```

A.6 Subroutine euler-n.f90

```
!////////////////////////////////////////////////////////////
!//                                                        //
!// Subroutine euler_n                                     //
!//                                                        //
!// Called by main program: nonlinear_swe                  //
!//                                                        //
!// Purpose:                                               //
!//                                                        //
!// To update thickness and volume transports for the first //
!// time step to avoid initial problem in CTCS-N           //
!//                                                        //
!////////////////////////////////////////////////////////////

subroutine euler_n(U,V,h,eta,x,y,alf,Hl,Hr,jmax,kmax, &
                   dx,dy,dt,iold,icur,nt,ntime,ibc,iev, &
                   f,g,A,eta0,H0,x0,y0)
  implicit none
!-------------------------------------------------------------
! Front matter
  ! Declarations
  integer, intent(in)           :: jmax,kmax,iold,icur,nt,ntime
  integer, intent(in)           :: ibc,iev
  real, intent(in)              :: dx,dy,dt,f,g,A,eta0,H0,x0,y0
  integer                       :: j,k,n,jp1
  real, dimension(jmax,kmax,nt) :: U,V,h,eta
  real, dimension(kmax)         :: corV,Hl,Hr,y
  real, dimension(jmax)         :: alf,x
  real                          :: AUx,AUy,AVx,AVy,MU,MV
  real                          :: mixu,mixv,R,corU
  real                          :: CU,CV,PU,PV
  real                          :: dt2,ax,ay,pru,prv,a4x,a4y
  real                          :: Uhjpk,Uhjk,UAjk,VAjk,hAjk
```

```
  real                          :: UAjkm,VAjkm,hAjkmUAjmk,VAjmk
  real                          :: hAjmk,Vhjkp,Vhjk

  ! Compute constants needed
  ax   = dt/dx
  ay   = dt/dy
  pru  = 0.5*g*ax
  prv  = 0.5*g*ay
  a4x  = 0.25*ax
  a4y  = 0.25*ay
  mixu = iev*dt*A/(dx*dx)
  mixv = iev*dt*A/(dy*dy)

! End front matter
!----------------------------------------------------------------
! Update variables
  ! 1: h and eta
  do k=2,kmax-1
     do j=2,jmax
        ! Update h
        h(j,k,icur) = h(j,k,iold) - &
                  ax*( U(j,k,iold) - U(j-1,k,iold) ) - &
                  ay*( V(j,k,iold) - V(j,k-1,iold) )
        ! Update eta
        eta(j,k,icur) = h(j,k,icur) - H0
     end do
     if(ibc == 1) then               ! Cyclic boundary condition
        h(1,k,icur) = h(jmax,k,icur)
        eta(1,k,icur) = eta(jmax,k,icur)
     else
        do j=1,jmax                  ! FRS boundary condition
           if(x(j) <= x0) then
              h(j,k,icur) = ( 1. - alf(j) )*h(j,k,icur) + &
                                    alf(j)*Hl(k)
```

```
         else
            h(j,k,icur) = ( 1. - alf(j) )*h(j,k,icur) + &
                                    alf(j)*Hr(k)
         endif
         eta(j,k,icur)   = h(j,k,icur) - H0
      end do
   endif
end do

! 2: U
do k=1,kmax
   corV(k) = 0.25*f*dt
end do
corV(2)  = 0.5*f*dt
corV(kmax-1) = 0.5*f*dt
corV(kmax) = 0.5*f*dt
do k=2,kmax-1
   if(ibc == 1) then               ! Cyclic boundary condition
      do j=2,jmax
         ! Recall cyclic boundary condition east-west => all
         ! references to j+1 must be replaced by 2 when
         ! j = jmax
         if(j == jmax) then ! Cyclic bc
            jp1 = 2
         else
            jp1 = j+1
         endif
         ! Coriolis term
         CU    = corV(k)*( V(j,k-1,iold) + V(j,k,iold) + &
                           V(jp1,k,iold) + V(jp1,k-1,iold) )
         ! Pressure term
         PU = - pru*( h(jp1,k,iold) + h(j,k,iold) )* &
                    ( eta(jp1,k,iold) - eta(j,k,iold) )
         ! Advective, non-linear flux terms
```

```
      Uhjpk  = U(jp1,k,iold) + U(j,k,iold)
      VAjk   = V(jp1,k,iold) + V(j,k,iold)
      Uhjk   = U(j,k,iold) + U(j-1,k,iold)
      UAjk   = U(j,k+1,iold) + U(j,k,iold)
      UAjkm  = U(j,k,iold) + U(j,k-1,iold)
      VAjkm  = U(jp1,k-1,iold) + U(j,k-1,iold)
      if(k == 2) then
         hAjkm = 2.*( h(j,k,iold) + h(j+1,k,iold) )
      else
         hAjkm = h(j,k-1,iold) + h(j,k,iold) + &
                 h(jp1,k,iold) + h(jp1,k-1,iold)
      endif
      if(k == kmax-1) then
         hAjk  = 2.*( h(j,k,iold) + h(jp1,k,iold) )
      else
         hAjk  = h(j,k,iold) + h(j,k+1,iold) + &
                 h(jp1,k+1,iold) + h(jp1,k,iold)
      endif
      AUx  = - a4x*( ( (Uhjpk*Uhjpk)/h(jp1,k,iold) ) - &
                     ( ( Uhjk*Uhjk )/h(j,k,iold) ) )
      AUy  = - ay*( ( (UAjk*VAjk  )/hAjk ) - &
                    ( ( UAjkm*VAjkm )/hAjkm ) )
      ! Mixing terms (ordinary second order)
      MU = mixu*( U(jp1,k,iold) - 2.*U(j,k,iold) + &
                                  U(j-1,k,iold) ) + &
           mixv*( U(j,k+1,iold) - 2.*U(j,k,iold) + &
                                  U(j,k-1,iold) )
      ! Update U
      U(j,k,icur) = U(j,k,iold) + CU + PU + AUx + AUy + MU
   end do
else
   do j=1,jmax                    ! FRS boundary conditions
      U(j,k,icur) = ( 1. - alf(j) )*U(j,k,icur) + alf(j)*0.
   end do
```

```
   endif
end do
! Impose closed boundary conditions
do j=1,jmax
 U(j,1,icur) = - U(j,2,icur)
 U(j,kmax,icur) = - U(j,kmax-1,icur)
end do

! 3: V
corU = 0.25*f*dt
do k=2,kmax-1
   if(ibc == 1) then               ! Cyclic boundary condition
      do j=2,jmax
         ! Recall cyclic boundary condition east-west => all
         ! references to j+1 must be replaced by 2 when
         ! j = jmax
         if(j == jmax) then ! Cyclic bc
            jp1 = 2
         else
            jp1 = j+1
         endif
         ! Coriolis term
         CV = - corU*( U(j-1,k,iold) + U(j-1,k+1,iold) + &
                       U(j,k+1,iold) + U(j,k,iold) )
         ! Pressure
         PV = - prv*( h(j,k+1,iold) + h(j,k,iold) )* &
                    ( eta(j,k+1,iold) - eta(j,k,iold) )
         ! The advective, non-linear flux terms
         UAjk  = U(j,k+1,iold) + U(j,k,iold)
         VAjk  = V(jp1,k,iold) + V(j,k,iold)
         hAjk  = h(j,k,iold) + h(j,k+1,iold) + &
                 h(jp1,k+1,iold) + h(jp1,k,iold)
         UAjmk = U(j-1,k+1,iold) + U(j-1,k,iold)
         VAjmk = V(j,k,iold) + V(j-1,k,iold)
```

```
            hAjmk = h(j-1,k,iold) + h(j-1,k+1,iold) + &
                    h(j,k+1,iold) + h(j,k,iold)
            Vhjkp = V(j,k+1,iold) + V(j,k,iold)
            Vhjk  = V(j,k,iold) + V(j,k-1,iold)
            AVx   = - ax*( ( UAjk*VAjk )/hAjk - &
                           ( UAjmk*VAjmk )/hAjmk )
            AVy   = - a4y*( ( Vhjkp*Vhjkp )/h(j,k+1,iold) - &
                            ( Vhjk*Vhjk )/h(j,k,iold) )
            ! Mixing terms
            MV = mixu*( V(jp1,k,iold) - 2.*V(j,k,iold) + &
                                        V(j-1,k,iold) ) + &
                 mixv*( V(j,k+1,iold) - 2.*V(j,k,iold) + &
                                        V(j,k-1,iold) )
            ! Update V
            V(j,k,icur) = V(j,k,iold) + CV + PV + AVx + AVy + MV
end do
     else                              ! FRS boundary conditions
        do j=1,jmax
           V(j,k,icur) = ( 1. - alf(j) )*V(j,k,icur) + alf(j)*0.
        end do
     endif
  end do
  ! Closed boundary conditions
  do j=1,jmax
   V(j,1,icur) = 0.
   V(j,kmax-1,icur) = 0.
  end do

! End updating variables
!---------------------------------------------------------------

  return

end subroutine euler_n
```

A.7 Subroutine ctcs-l.f90

```
!////////////////////////////////////////////////////////////
!//                                                        //
!// Subroutine ctcs_l                                      //
!//                                                        //
!// Called by main program: nonlinear_swe                  //
!//                                                        //
!// Purpose:                                               //
!//                                                        //
!// Updates thickness and transports using the linear CTCS //
!// scheme. First time step uses the subroutine Euler_linear. //
!//                                                        //
!////////////////////////////////////////////////////////////

subroutine ctcs_l(U,V,h,eta,x,y,alf,Hl,Hr,jmax,kmax,iold,icur, &
                  inew,nt,ntime,iev,ibc,dx,dy,dt,dt2, &
                  f,g,A,eta0,H0,x0,y0)
  implicit none
!-------------------------------------------------------------
! Front matter
  ! Declarations
  integer, intent(in)           :: jmax,kmax,iold,icur,inew
  integer, intent(in)           :: nt,ntime,iev,ibc
  real, intent(in)              :: dx,dy,dt,dt2,f,g,A,eta0
  real, intent(in)              :: H0,x0,y0
  integer                       :: j,k,n,jp1
  real, dimension(jmax,kmax,nt) :: U,V,h,eta
  real, dimension(kmax)         :: corV,Hl,Hr,y
  real, dimension(jmax)         :: alf,x
  real                          :: CU,CV,PU,PV,MU,MV
  real                          :: ax,ay,pru,prv,a4x,a4y
  real                          :: mixu,mixv,R,corU
```

```
  ! Compute needed constants
  ax   = dt2/dx
  ay   = dt2/dy
  pru  = g*H0*ax
  prv  = g*H0*ay
  mixu = iev*dt2*A/(dx*dx)
  mixv = iev*dt2*A/(dy*dy)
  R = 1./( 1. + mixu + mixv )
!
! End front matter
!----------------------------------------------------------------
! Update variables
  ! 1; h and eta
  do k=2,kmax-1
     do j=2,jmax
        ! Update h
        h(j,k,inew) = h(j,k,iold) - &
                  ax*( U(j,k,icur) - U(j-1,k,icur) ) - &
                  ay*( V(j,k,icur) - V(j,k-1,icur) )
        ! Update eta
        eta(j,k,inew) = h(j,k,inew) - H0
     end do
     if(ibc == 1) then               ! Cyclic boundary condition
        h(1,k,inew) = h(jmax,k,inew)
        eta(1,k,inew) = eta(jmax,k,inew)
     else
        do j=1,jmax                  ! FRS boundary condition
           if(x(j) <= x0) then
              h(j,k,inew) = ( 1. - alf(j) )*h(j,k,inew) + &
                                    alf(j)*Hl(k)
           else
              h(j,k,inew) = ( 1. - alf(j) )*h(j,k,inew) + &
                                    alf(j)*Hr(k)
           endif
```

```
            eta(j,k,inew)   = h(j,k,inew) - H0
         end do
      endif
   end do

   ! 2: U
   do k=1,kmax
corV(k) = 0.5*f*dt
   end do
   corV(2)      = f*dt
   corV(kmax-1) = f*dt
   corV(kmax)   = f*dt
   do k=2,kmax-1
      if(ibc == 1) then               ! Cyclic boundary condition
         do j=2,jmax
            ! Recall cyclic boundary condition east-west => all
            ! references to j+1 must be replaced by 2 when
            ! j = jmax
            if(j == jmax) then
               jp1 = 2
            else
               jp1 = j+1
            endif
            ! Coriolis term U
            CU = corV(k)*( V(j,k-1,icur) + V(j,k,icur) + &
                           V(jp1,k,icur) + V(jp1,k-1,icur) )
            ! Pressure term U
            PU = - pru*( eta(jp1,k,icur) - eta(j,k,icur) )
            ! Mixing terms U (Dufort-Frankel)
            MU = mixu*( U(jp1,k,icur) - U(j,k,iold) + &
                                        U(j-1,k,icur) ) &
               + mixv*( U(j,k+1,icur) - U(j,k,iold) + &
                                        U(j,k-1,icur) )
            ! Update U
```

```
            U(j,k,inew) = ( U(j,k,iold) + CU + PU + MU )*R
end do
         U(1,k,inew) = U(jmax,k,inew)
      else
         do j=1,jmax                    ! FRS boundary condition
            U(j,k,inew) = ( 1. - alf(j) )*U(j,k,inew) + alf(j)*0.
         end do
      endif
   end do
   ! Closed boundary conditions U
   do j=1,jmax
U(j,1,inew) = - U(j,2,inew)
U(j,kmax,inew) = - U(j,kmax-1,inew)
   end do

! 3: V
   corU = 0.5*f*dt
   do k=2,kmax-1
      if(ibc == 1) then                 ! Cyclic boundary condition
         do j=2,jmax
            ! Recall cyclic boundary condition east-west => all
            ! references to j+1 must be replaced by 2 when
            ! j = jmax
            if(j == jmax) then
               jp1 = 2
            else
               jp1 = j+1
            endif
            ! Coriolis term V
            CV    = - corU*( U(j-1,k,icur) + U(j-1,k+1,icur) + &
                             U(j,k+1,icur) + U(j,k,icur) )
            ! Pressure term V
            PV    = - prv*( eta(j,k+1,icur) - eta(j,k,icur) )
            ! Mixing terms V (Dufort-Frankel)
```

```
            MV   = mixu*( V(jp1,k,icur) - V(j,k,iold) + &
                                          V(j-1,k,icur) ) + &
                   mixv*( V(j,k+1,icur) - V(j,k,iold) + &
                                          V(j,k-1,icur) )
            ! Update V
            V(j,k,inew) = ( V(j,k,iold) + CV + PV + MV )*R
end do
        V(1,k,inew) = V(jmax,k,inew)
     else
        do j=1,jmax                  ! FRS boundary condition
        V(j,k,inew) = ( 1. - alf(j) )*V(j,k,inew) + alf(j)*0.
        end do
     endif
  end do
  ! Closed boundary condition V
  do j=1,jmax
     V(j,1,inew) = 0.
     V(j,kmax-1,inew) = 0.
  end do

! End updating variables
!---------------------------------------------------------------

  return

end subroutine ctcs_l
```

A.8 Subroutine euler-l.f90

```
!////////////////////////////////////////////////////////////
!//                                                        //
!// Subroutine euler_l                                     //
!//                                                        //
!// Called by main program: nonlinear_swe                  //
!//                                                        //
!// Purpose:                                               //
!//                                                        //
!// To update variables using the linear version of the Euler //
!// scheme.                                                //
!//                                                        //
!////////////////////////////////////////////////////////////

subroutine euler_l(U,V,h,eta,x,y,alf,Hl,Hr,jmax,kmax, &
                   iold,icur,nt,ntime,iev,ibc,dx,dy,dt, &
                   f,g,A,eta0,H0,x0,y0)
  implicit none
!-------------------------------------------------------------
! Front matter
  ! Declarations
  integer, intent(in)           :: jmax,kmax,iold,icur
  integer, intent(in)           :: nt,ntime,iev,ibc
  real, intent(in)              :: dx,dy,dt,f,g,A,eta0,H0,x0,y0
  integer                       :: j,k,n,jp1
  real, dimension(jmax,kmax,nt) :: U,V,h,eta
  real, dimension(kmax)         :: corV,Hl,Hr,y
  real, dimension(jmax)         :: alf,x
  real                          :: CU,CV,PU,PV,MU,MV,fdt
  real                          :: dt2,ax,ay,pru,prv,a4x
  real                          :: a4y,mixu,mixv,R,corU

  ! Compute factor in front of the Coriolis terms
```

```
  fdt = f*dt

  ! Compute needed constants
  ax   = dt/dx
  ay   = dt/dy
  pru  = g*H0*ax
  prv  = g*H0*ay
  mixu = iev*dt*A/(dx*dx)
  mixv = iev*dt*A/(dy*dy)

! End front matter
!----------------------------------------------------------------
! Update variables
  ! 1: h and eta
  do k=2,kmax-1
     do j=2,jmax
        ! Update h
        h(j,k,icur) = h(j,k,iold) - &
                      ax*( U(j,k,iold) - U(j-1,k,iold) ) - &
                      ay*( V(j,k,iold) - V(j,k-1,iold) )
        ! Update eta
        eta(j,k,icur) = h(j,k,icur) - H0
     end do
     if(ibc == 1) then ! Cyclic boundary condition
        h(1,k,icur) = h(jmax,k,icur)
        eta(1,k,icur) = eta(jmax,k,icur)
     else
        do j=1,jmax ! FRS boundary condition
           if(x(j) <= x0) then
              h(j,k,icur) = ( 1. - alf(j) )*h(j,k,icur) + &
                                       alf(j)*Hl(k)
           else
              h(j,k,icur) = ( 1. - alf(j) )*h(j,k,icur) + &
                                     alf(j)*Hr(k)
```

```
            endif
            eta(j,k,icur) = h(j,k,icur) - H0
         end do
      endif
   end do

   ! 2: U
   do k=1,kmax
corV(k) = 0.25*fdt
   end do
   corV(2)  = 0.5*fdt
   corV(kmax-1) = 0.5*fdt
   corV(kmax) = 0.5*fdt

   do k=2,kmax-1
      if(ibc == 1) then                ! Cyclic boundary condition
         do j=2,jmax
            ! Recall cyclic boundary condition east-west => all
            ! references to j+1 must be replaced by 2 when
            ! j = jmax
            if(j == jmax) then
               jp1 = 2
            else
               jp1 = j+1
            endif
            ! Coriolis term U
            CU = corV(k)*( V(j,k-1,iold) + V(j,k,iold) + &
                           V(jp1,k,iold) + V(jp1,k-1,iold) )
            ! Pressure term U
            PU = - pru*( eta(jp1,k,iold) - eta(j,k,iold) )
            ! Mixing terms (ordinary second order)
            MU = mixu*( U(jp1,k,iold) - 2.*U(j,k,iold) + &
                                        U(j-1,k,iold) ) + &
                 mixv*( U(j,k+1,iold) - 2.*U(j,k,iold) + &
```

```
                                          U(j,k-1,iold) )
            ! Update U
            U(j,k,icur) = U(j,k,iold) + CU + PU + MU
         end do
         U(1,k,icur) = U(jmax,k,icur)
      else
         do j=1,jmax                    ! FRS boundary condition
            U(j,k,icur) = ( 1. - alf(j) )*U(j,k,icur) + alf(j)*0.
         end do
      endif
   end do
   ! Impose closed boundary conditions on U
   do j=1,jmax
U(j,1,icur) = - U(j,2,icur)
U(j,kmax,icur) = - U(j,kmax-1,icur)
   end do

   ! 3: V
   corU = 0.25*fdt
   do k=2,kmax-1
      if(ibc == 1) then               ! Cyclic boundary condition
         do j=2,jmax
            ! Recall cyclic boundary condition east-west => all
            ! references to j+1 must be replaced by 2 when
            ! j = jmax
            if(j == jmax) then
               jp1 = 2
            else
               jp1 = j+1
            endif
            ! Coriolis term V
            CV = - corU*( U(j-1,k,iold) + U(j-1,k+1,iold) + &
                          U(j,k+1,iold) + U(j,k,iold) )
            ! Pressure term V
```

```
            PV = - prv*( eta(j,k+1,iold) - eta(j,k,iold) )
            ! Mixing terms (ordinary second order)
            MV = mixu*( V(jp1,k,iold) - 2.*V(j,k,iold) + &
                                        V(j-1,k,iold) ) + &
                 mixv*( V(j,k+1,iold) - 2.*V(j,k,iold) + &
                                        V(j,k-1,iold) )
            ! Update V
            V(j,k,icur) = V(j,k,iold) + CV + PV + MV
end do
        V(1,k,icur) = V(jmax,k,icur)
     else                              ! FRS boundary condition
        do j=1,jmax
           V(j,k,icur) = ( 1. - alf(j) )*V(j,k,icur) + alf(j)*0.
        end do
     endif
  end do
  ! Impose closed boundary conditions on V
  do j=1,jmax
V(j,1,icur) = 0.
V(j,kmax-1,icur) = 0.
  end do

! End updating variables
!---------------------------------------------------------------

  return

end subroutine euler_l
```

A.9 Subroutine `fbl.f90`

```
!////////////////////////////////////////////////////////////
!//                                                        //
!// Subroutine fbl                                         //
!//                                                        //
!// Called by main program: nonlinear_swe                  //
!//                                                        //
!// Purpose:                                               //
!//                                                        //
!// Updates thickness and transports using the Sielecki    //
!// (Forward-Backward) scheme.                             //
!//                                                        //
!////////////////////////////////////////////////////////////

subroutine fbl(U,V,h,eta,x,y,alf,Hl,Hr,jmax,kmax,iold,inew, &
               nt,ntime,ibc,dx,dy,dt,f,g,H0,x0,y0)

  implicit none
!-------------------------------------------------------------
! Front matter
  ! Declarations
  integer, intent(in)           :: jmax,kmax,iold,inew
  integer, intent(in)           :: nt,ntime,ibc
  real, intent(in)              :: dx,dy,dt,f,g,H0,x0,y0
  integer                       :: j,k,n,jp1
  real, dimension(jmax,kmax,nt) :: U,V,h,eta
  real, dimension(kmax)         :: corV,Hl,Hr,y
  real, dimension(jmax)         :: alf,x
  real                          :: CU,CV,PU,PV,fdt
  real                          :: dt2,ax,ay,pru,prv,a4x,a4y
  real                          :: mixu,mixv,R,corU
  real                          :: arg11,arg12
```

```
  ! Compute factor in front of the Coriolis terms
  fdt = f*dt

  ! Compute needed constants
  ax   = dt/dx
  ay   = dt/dy
  pru  = g*H0*ax
  prv  = g*H0*ay

! End front matter
!----------------------------------------------------------------
! Update variables
  ! Compute factor in front of the Coriolis terms
  corU = 0.25*fdt
  do k=1,kmax
corV(k) = 0.25*fdt
  end do
  corV(2)  = 0.5*fdt
  corV(kmax-1) = 0.5*fdt
  corV(kmax) = 0.5*fdt

  ! 1: h and eta
  do k=2,kmax-1
     do j=2,jmax
        ! Update h
        h(j,k,inew) = h(j,k,iold) - &
                      ax*( U(j,k,iold) - U(j-1,k,iold) ) - &
                      ay*( V(j,k,iold) - V(j,k-1,iold) )
        ! Update eta
        eta(j,k,inew) = h(j,k,inew) - H0
     end do
     if(ibc == 1) then               ! Cyclic boundary condition
        h(1,k,inew) = h(jmax,k,inew)
        eta(1,k,inew) = eta(jmax,k,inew)
```

```
      else
         do j=1,jmax                        ! FRS boundary condition
            if(x(j) <= x0) then
               h(j,k,inew) = ( 1. - alf(j) )*h(j,k,inew) + &
                                  alf(j)*Hl(k)
            else
               h(j,k,inew) = ( 1. - alf(j) )*h(j,k,inew) + &
                                  alf(j)*Hr(k)
            endif
            eta(j,k,inew) = h(j,k,inew) - H0
         end do
      endif
   end do
   ! 2: U
   do k=2,kmax-1
      if(ibc == 1) then                  ! Cyclic boundary condition
         do j=2,jmax
            ! Recall cyclic boundary condition east-west => all
            ! references to j+1 must be replaced by 2 when
            ! j = jmax
            if(j == jmax) then
               jp1 = 2
            else
               jp1 = j+1
            endif
            ! Coriolis term U
            CU = corV(k)*( V(j,k-1,iold) + V(j,k,iold) + &
                           V(jp1,k,iold) + V(jp1,k-1,iold) )
            ! Pressure term
            PU = - pru*( eta(jp1,k,inew) - eta(j,k,inew) )
            ! Update U
            U(j,k,inew) = U(j,k,iold) + CU + PU
end do
      else                               ! FRS boundary condition
```

```
        do j=1,jmax
           U(j,k,inew) = ( 1. - alf(j) )*U(j,k,inew) + alf(j)*0.
        end do
     endif
  end do
  ! Impose closed boundary conditions on U
  do j=1,jmax
     U(j,1,inew) = - U(j,2,inew)
     U(j,kmax,inew) = - U(j,kmax-1,inew)
  end do

  ! 3: V
  do k=2,kmax-1
     if(ibc == 1) then                ! Cyclic boundary condition
        do j=2,jmax
           ! Recall cyclic boundary condition east-west => all
           ! references to j+1 must be replaced by 2 when
           ! j = jmax
           if(j == jmax) then
              jp1 = 2
           else
              jp1 = j+1
           endif
           ! Coriolis term V
           CV = - corU*( U(j-1,k,inew) + U(j-1,k+1,inew) + &
                         U(j,k+1,inew) + U(j,k,inew) )
           ! Pressure
           PV  = - prv*( eta(j,k+1,inew) - eta(j,k,inew) )
           ! Update V
           V(j,k,inew) = V(j,k,iold) + CV + PV
end do
        V(1,k,inew) = V(jmax,k,inew)
     else                             ! FRS boundary condition
       do j=1,jmax
```

```
          V(j,k,inew) = ( 1. - alf(j) )*V(j,k,inew) + alf(j)*0.
        end do
      endif
  end do
  ! Impose closed boundary conditions on V
  do j=1,jmax
V(j,1,inew) = 0.
V(j,kmax-1,inew) = 0.
  end do

! End updating variables
!---------------------------------------------------------------------

  return

end subroutine fbl
```